Mathematical and Geometrical Formulation/Analysis for Beam Divergence Limit Angle in Radiotherapy Wedges

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Abstract: Wedge filters constitute a useful type of Beam Modification Devices (BMD) in Radiation Therapy. Its use is routinary in both Forward and Inverse Treatment Planning Optimization (ITPO). In previous contributions we presented the exact/approximated path of a Pencil Photon-Beam (AAA Model, Anisotropic Analytic Algorithm), through standard manufacturing alloy wedges. It was found a so-defined Limit-Angle (LA), beyond of which the outpoint of the beam is located improperly at the lateral side of the wedge. LA exists because of the photon-beam physical divergence phenomenon. In this paper we carry out the Geometrical and Analytical determination of the LA in function of the beam divergence angle, collimator output distance, and the size parameters of the wedge filter. Two methods are used, Geometrical and Analytical. Boundaries formulation is presented with inequalities. A series of Mathematical Formulations for LAs, is shown with basic approximations according to the industrial manufacturing wedge standards. In addition, an extension of these geometrical approximations is applied on so-called Conformal wedges (CW) [11.1, 11.2]. A primary stage for CW Limit Angle is determined. Formulation is verified with Optimization Mathematical Methods.

Keywords: AAA, ITPO, BMD, Static Wedge Filters, Nonlinear Optimization, Analytic Geometry, Industrial Manufacturing Standards (IMS), Conformal Wedge (CW).

I. INTRODUCTION

Wedge filters are commonly used in RT Treatment Planning Optimization (TPO), Forward and Inverse Methods. Its function is to conform the tumor shape, and avoid hot spots without excessive technical effort, usually for superficial cancer (e.g., Larynx, Breast, Lung, or Prostate tumors) [11, 11.1, 11.2]. Wedges belong to the generic group of Beam Modification Devices (BMD) [33], which constitutes a useful type of technical resources in TPO. It is not unusual to combine Multi Leaf Collimator modification with wedge filters (wedges, virtual wedges, universal wedges and other techniques), to carry out an optimal adaptation of the dose delivery within the Tumor Target. Among the recent techniques for wedge filters we find the omnifilter wedge [41].

In addition to the principal mechanical and physical tools and parts of the new accelerators, the complementary use of Beam Modification Devices (BMDs) are useful to get conformal dosage and avoid hot spots without excessive technical effort. The inconvenient of the BMDs is the dose distribution alteration, due mainly to the increment of scattering photons (more frequent in Kilo voltage than Megavoltage). BMD types are classified [Sharma, 2011] by Shielding, Compensation, Wedge Filtration and Flattering. There are a number of BMDs, among them; those most frequently used are Multileaf Collimators (MLC), Shielding, Compensators, Wedge Filters, and Penumbra Trimmers. We focus this paper on Wedge Filters, which are used in general for relatively superficial tumors. Wedges can be classified as: Universal, Dynamic, Virtual, and Pseudo-Wedges [33]. The use of wedges is justified for practical, technical and economic reasons. High-quality alloy materials are not expensive, and the size of the wedges makes their handling, change and substitution easy. Manufacturing of wedge filter series with different angles (usually 15°, 30°, 45°, 60°) is neither difficult nor expensive. Alloy is a high-endurance material which provides long industrial life for continuous RT sessions. In addition, several combinations of wedges in 2D and 3D configurations, the so-called omnifilter system, may be used to obtain a series of radiation distribution(s) to adapt the dose delivery to the tumor shape with accurate approximations and engineering precision.

Attenuation dose of the wedge (exponential factor, Eq [22]), depends mainly on the beam path through the wedge material, the material composition itself, and proportionality on the beam divergence angle [11]. Previously, [11], an exact/approximated geometrical calculation of this path was published for the AAA Model in 3D/2D, so-called Anisotropic Analytical Algorithm [3439], with acceptable RMS error values in general. These formulas depend on the beam divergence angle (BDA), among other parameters. Basic simulations were also shown to determine error attenuation factors within the integral dose formula of the AAA algorithm.

However, apart from the path length and geometrical approximations, we consider the path through the wedge as dependent also on the BDA [6-11]. This fact implies that BDAs could become a technical problem if the beam approaches the lateral borders, when passing throughout the filter. In such cases, it could happen that the outpoint of the
beam is improperly located at the lateral side of the wedge. Symmetrical distribution of the incident beam over the wedge surface, without approaching too much the wedge borders, is desirable for precision/accuracy in the Treatment Plan. It was found in simulation data, a so-defined Limit Angle (LA), beyond of which this inconvenient physical-geometrical phenomenon could occur [11].

Photon-Beam divergence angles values vary around 20 degrees. The Beam minimum divergence depends on the collimator design quality, and in general of the precision engineering manufacturing of the LINACs.

In this technical paper we present a series of geometrical and analytical formulas for LA determination in standard wedge filters. Mathematical developments were tried to be shown clear, complete, and understandable. In addition, coordinate systems could be used/modified for other Radiotherapy Dosimetry Models, apart from the classical AAA algorithm [35, 36].

II. LIMIT ANGLE FOR GENERAL DIVERGENCE ANGLE

This Section deals with the methods used to define mathematically the LA from a geometrical and analytical point of analysis.

1. Algebraic Geometry Approximations and Geodesics

In Fig 1 we draw the line that joints P (LINAC collimator output) with the inferior wedge border and intersects the axis $u_2$ at $u_1=0$. The distance $r$ at $u_2$ axis by similar triangles is given by similar triangles, since the equation of the inferior plane [11] is

$$z = t g a (u_1 + a);$$

[Ref 11]

therefore

$$\frac{P + z}{b} = \frac{P}{r}.$$  \[Equation 1\]

then,

$$r = \frac{(b \cdot P)}{(P + z)} = \frac{(b \cdot P)}{(P + t g a \cdot a)};$$ \[Eq 2\]

Now we rotate the equation of the line $P-r$ along the superior wedge surface 90 degrees to reach the axis $u_1$ (Fig 2) when $u_2=0$. If so, there is an inferior geodesic at the surface of the cylinder of b radius (Fig 3). This inferior geodesic is defined by the intersection of cylinder and wedge inferior plane (Fig 3). The lowest value of $\theta$ corresponds to the values [11.1],

$$u_2 = 0; u_1 = r_{u_2=0}$$  \[Eq 3\]

Since the radius of cylinder is b, at that point we have,

$$z = t g a (b + a);$$

[Eq 4]

Therefore, this minimum angle is the LA, because if we take any other higher value along the inferior geodesic, the angle would be higher, and the beam output could go beyond the inferior geodesic. This angle is useful for any value of $u_1$ and $u_2$. So we get

$$r_{u_2=0} = b \cdot P \left[ \frac{1}{(t g a \cdot (b + a) + P)} \right]$$\[Eq 5\]

then,

$$t g a (\theta_L) = \left( \frac{r_{u_2=0}}{P} \right);$$ \[Eq 6\]

and the resulting formula is

$$\theta_L \left[ Geometrical \right] = arct g \left( \frac{r_{u_2=0}}{P} \right);$$

[Eq 7]

According to Figs 1,2,3, in general, the radius of the superior wedge surface geodesic is by trigonometry,

$$\frac{b}{r} = \frac{(t g a \cdot (u_1 + a) + P)}{P};$$

or

$$r = b \cdot P \cdot \left[ \frac{1}{(P + t g a \cdot (u_1 + a))} \right];$$ \[Eqs 8\]

The subsequent step is to define the equations of the geodesic that set the limit angle over the superior surface of the wedge. We will define a geodesic which guarantees the previous defined $\theta_L$ [Geometrical] holds, and a general curve described by the conditions of Figs 1 and 2. Taking the length of the distance at $u_2$ defined in Fig 2, we get

$$\frac{P}{r} = \frac{(P + t g a \cdot (u_1 + a))}{b};$$

then

$$r = b \cdot P \cdot \left[ \frac{1}{(P + t g a \cdot (u_1 + a))} \right];$$ \[Eqs 9\]
This circle over the superior surface makes sure that the pencilbeam is correctly conducted. But it is also useful to define \( r \) in function of \( \theta \), such as

\[
\tan \theta = \frac{r}{p};
\]

[Eq 10]

then from the previous expression

\[
r = b - \tan \theta \cdot [\tan \alpha \cdot (b + a)] = r(\theta);
\]

[Eq 11]

If we follow the same method with the data of Figs 1 and 2, the curve described by the beam whose output is at inferior geodesic of Fig 3 is given by the equation

\[
r = b - \tan \theta \cdot \left[ \tan \alpha \cdot \left( \sqrt{(b^2 - u_2^2)} + a \right) \right] = r(\theta);
\]

[Eq 12]

and we have used the constraint for inferior geodesic

\[
u_1^2 + u_2^2 = b^2;
\]

[Eq 13]

\[
\frac{\partial f(u_2)}{\partial u_2} = 0
\]

\[
\frac{\partial^2 f(u_2)}{\partial u_2^2} \neq 0
\]

[Eq 16]

We find that for \( u_1 = b \) this condition is held. What has been done is to prove the optimal LA both geometrically and analytically. In IMRT it is practical to carry out a beamlet-discretization to set the angle limits more accurately. This method yields a refinement in the LA upper boundaries. The straight lines that define the lower boundaries of the wedge volume are two. One lateral with the same gradient of the inferior plane of the wedge and other perpendicular to this largest wall zone which is the border of the broad part of the wedge. The equations of the straight lines are not complicated to be determined through analytic geometry. In that way, the distances among the collimator output \( P \), beamlet by beamlet to discretized points of these lines can be also calculated. We refer this formulation to subsequent publications, and set this initial point for dose delivery precision in software/planning IMRT. Now we proceed to apply the same concept on the previous [11] method of beam divergence angle decomposition.

2.-ANALYTICAL MINIMIZATION. IMRT Voxel-Beamlet Discretization/Approximation.

The analytical proof that \( \theta_L \) [geometrical] is a minimum is based on the fact that when \( L \) (Figs, 1,2,3), is maximum, \( \theta_L \) is minimum. The output of the beam path is always at or into the geodesics of the cylinder of radius \( b \). Then, at any point of the cylinder geodesic

\[
u_1^2 + u_2^2 = b^2;
\]

[Eq 14]

and

\[
L^2 = f(u_2) = h^2 + \left( P + \tan \alpha \cdot \left( \sqrt{(h^2 - u_2^2)} + a \right) \right)^2;
\]

[Eq 15]

If we maximize this function, we find a maximum for \( u_1 = b \), with the second derivative for this value positive. It is necessary to consider in this maximization that \( P \) is in the negative direction of axis \( Z \). Then, the result of maximum for \( u_1 = b \) holds. Therefore, we have proven that this LA [geometrical] is minimum and applicable to all directions to avoid output of beam at lateral sides of the wedge. That is, the maximization is,

\[
\frac{(P + 2c)}{a} = \frac{P}{r_1};
\]

that is

\[
r_1 = \frac{(a \cdot P)}{(P + 2c)};
\]

or

\[
tg\theta_1 = \frac{r_1}{P} = \frac{a}{(P + 2c)};
\]

with

\[
\theta_1 = arctg \left( \frac{a}{(P + 2c)} \right);
\]

now with \( \theta_2 \),

\[
\frac{(P + c)}{a} = \frac{P}{r_2};
\]

\[
\frac{(P + c)}{a} = \frac{P}{r_2};
\]
\[ \tan \theta_2 = \left[ \frac{(P + c)}{a} \right]^{-1} = \left[ \frac{P}{r_2} \right]^{-1}; \]

Therefore

\[ \theta_2 = \arctg \left[ \frac{a}{(P + c)} \right]; \]

[Eqs 17]

Therefore, to make sure the components of the decomposed beam have a correct output point the following conditions should hold

\[ \theta_1 \leq \arctg \left[ \frac{a}{(P + 2c)} \right]; \]

and

\[ \theta_2 \leq \arctg \left[ \frac{a}{(P + c)} \right]; \]

[Eqs 18]

Then, we have determined the mathematical and geometrical conditions when using beam-decomposition angles. Any beam whose decomposed angles accomplish this is confined into the right path when emerging from the inferior part of the wedge. We step forward to explain another approximation calculated at the inferior plane of the wedge, Fig 8, taking the coordinates of the emerging beam/beamlet. We get \( r \) as beamlet position of the emergent beam in this quadrant, such as,

\[ r^2 = r_1^2 + r_2^2 \leq b^2; \]

then,

\[ \frac{(P \cdot r_1)}{(P + 2c)}; \]

and

\[ \frac{(P \cdot r_2)}{(P + 2c)}; \]

and we decompose the beam/beamlet angle,

\[ \tan \theta_2 = \frac{u_2}{P}; \]

\[ \tan \theta_1 = \frac{u_1}{P}; \]

\[ \tan \theta = \frac{r}{(P + 2c)}; \]

Hence we get,

\[ \frac{(P + 2c)^2}{P^2} \cdot \left[ \frac{p^2 \tan \theta_1^2 + p^2 \tan \theta_2^2}{b^2}; \right] \leq \frac{b^2}{(P + 2c)^2}; \]

or

\[ \tan^2 \theta = \tan^2 \theta_1 + \tan^2 \theta_2 \leq \frac{b^2}{(P + 2c)^2}; \]

[Eqs 18.1]

And these formulas are also useful for complementary determinations both for theoretical dose delivery and planning system. We refer the 3D formula defined in [11, an example in Table 1] for wedge filters path using decomposition is

\[ D^3 = \left[ u_1 - \frac{\tan \theta_1 (\sin \alpha)}{\tan \theta_1 \sin \alpha - \cos \alpha} \right]^2 + \]

\[ u_{2} - \frac{\tan \theta_2 (\sin \alpha)}{\tan \theta_1 \sin \alpha - \cos \alpha} \]

[Eq 19]

and the approximation for small angles [11]

\[ D^3 \approx \left[ u_1 - \frac{\theta_1 (\sin \alpha)}{\theta_1 \sin \alpha - \cos \alpha} \right]^2 + \]

\[ u_{2} - \frac{\theta_2 (\sin \alpha)}{\theta_1 \sin \alpha - \cos \alpha} \]


[Eq 20. From Ref 35]

We explain the physical-mathematical significance of the convolution within the integral dose for better learning. If we fix a point \((x, y)\), the exponential becomes maximum (equal to unity) when the \( u, v \) coordinates take that same value (straight direction, maximum quantity of photons). And the minimum value of the exponential is when the \( u, v \) values
have the same sign and highest value (opposite part of the beam and minimum fluence of photons coming from that zone).

where with a beam of cross-section 2a x 2b and

\[
\alpha' = \alpha \cdot \left(1 + \frac{z}{F}\right) ;
\]

[Fig 6]

and

\[
b' = b \cdot \left(1 + \frac{z}{F}\right) ;
\]

the fluence approximation for wedge filters from Ulmer and Harder (1996),

\[
\Phi_w(u,v,z) = \Phi_u(u,v,z) \cdot \exp\left[-\mu_w \left(L \pm \frac{cu}{F + z} \sin \alpha \right) \right] ;
\]

and the classical fluence equation developed by Ulmer and Harder after integration of erf functions (1996),

\[
\Phi_w(u,v,z) = \Phi_u(u,v,z) \cdot A \exp(-2qu) ;
\]

with

\[
A = \exp\left(-\mu_w L \frac{\sin \alpha}{\cos \alpha + \phi}\right) \text{ and}
\]

\[
2q = \pm \left(\frac{\mu_w c}{F + z} \sin \alpha \right) \frac{\cos \alpha + \phi}{\cos \alpha + \phi} ;
\]

The result of these transformations is the first exponential in [Eq 23. From Ref 35]. This is the reason to explain the variations at erf functions of x in [Eq 23. From Ref 35]. The angle \(\phi\) in [eq 21 From Ref 35.] is equivalent to theta angle in Figs 1-4. Alpha in [Eq 21. From Ref 35] is also the wedge angle. In [Eq 21. From Ref 35], note that if we treat this problem in 2D, there is always an error depending on coordinate v (through the plane of the image). In [Eq 21. From Ref 35] c is equivalent to P in our 3D formula [Figs 1-3, and 6], and z coordinate origin is taken lower at a distance P-C from the superior plane of the wedge and towards the patient surface. The distance L corresponds to the distance a in Figs 1-6. With these equivalences it is possible to set mathematical links between one model and the other.

**III. APPLICATIONS FOR CONFORMAL WEDGES**

In previous contributions [9,10,11], the LA was mathematically defined and developed for wedges. We detail here the main formulas and one sketch of LA, together with a picture of the so-called Conformal Wedge [11.1, 11.2]. A Conformal Wedge Filter has a sloping geometry divided into several non-continuous steps. The dose distribution in these types of wedges changes its shape for a more conformal radiation distribution, if the tumor presents irregular geometry/contour, rather non-spherical. The Conformal Radiotherapy Wedge was mathematically/physically designed by F Casesnoves (July 2005, Madrid City). Computational/Numerical Simulations were carried out at Denver, October 2012 (Patent in Pending Process). The wedge filter function is to attenuate the radiation beam in increasing magnitude, usually along the transversal direction to the photon-
beam. As a result, the dose delivery magnitude forms a curved distribution in that transversal direction for each radiation-depth value within the photon dose-deposition region. Classical wedges geometry have a straight sloping face corresponding to the hypotenuse of the triangle defined by the lateral sides. As was detailed previously, givien a fixed collimator output to wedge surface distance, LA is defined as the maximum angle of divergence that can be reached by the whole radiation beam without emerging at any point of lateral walls of the wedge. Photon-Beam divergence angles values vary around 20 degrees. The Beam minimum divergence depends on the collimator design quality, and in general of the precision engineering manufacturing of the LINACs. LA in conformal wedges is useful because of several reasons. Avoids hot spots, sub-optimal dose delivery, planning system software propagation errors, overdose at OARS, and repetition of planning work caused by sub-optimal dose delivery calculations. The LA for a conformal wedge calculation presents some additional difficulties. However, the primary approximation is to take as LA for a Conformal Wedge the value of the deepest step of the wedge. In Fig 7 we show a basic sketch of a conformal wedge.

IV. COMPUTATIONAL SOFTWARE

Several subroutines of Optimization Methods were used to check the accuracy of the mathematical formulation (Freemat, GNU, General Public License,Samit Basu). In particular, for geodesics curves we used polynomial subroutines. The graphics were useful to obtain imaging representations of the algorithms developed, plotting the path length related to divergence angle.

V. RESULTS AND FORMULATION

We have shown a summary of formulas useful, in general, for 15, 30, 45, and 60 standard wedge filters. Collimator output distance to wedge surface is given by variable P. We consider the results as an initial approximation in order to obtain more applicable/evolutioned algorithms for planning precision. Other use of the formulation could be focused on MLC techniques with/without wedges, static or dynamic. Conformal Wedges Formulation will be developed in further contributions [Fig 7]. All in all, formulas were made in a practical sense for cancer RT treatment planning.

VI. DISCUSSION AND CONCLUSIONS

Limit Angle data for Photon–Beam Divergence determination is useful when using wedge filters for conformal dose delivery. LA mathematical formulation is especially practical for high-divergent or poorly-collimated beams, and rather long-distance from collimator output to wedge filters (or any BMD of similar geometrical attenuation in general). If the beam is located properly over the wedge surface, the possibility to create hot spots in dose delivery can be reduced. LA is useful because of several reasons. Avoids hot spots and overdose, sub-optimal dose delivery, planning system software propagation errors, overdose at OARS, and repetition of planning work caused by sub-optimal dose delivery calculations.

The mathematical formulation/approximations presented can be considered a primary acceptable stage for overcoming this kind of technical problems in Radiation Therapy Inverse Treatment Planning, improving dose delivery optimization in superficial tumors (lung, breast, etc) and avoiding hot spots. LA formulation could also be appropriate for computational design of Planning Systems and Inverse/Forward Optimization Software. Finally, we presented a primary approximation in LA for Conformal Wedges (Casenoves, 2005).

Therefore, the mathematical development was done to set useful boundaries for confining the beam/IMRT-beam lets within the functional part of the wedges [Figs, 1,2,3,4]. In addition, we set boundaries for beam-decomposition angles, and related these limits with the principal divergence angles. What is more, it was explained the mathematical equivalence between the classical AAA/2D wedge formulation and the 3D equations that have been developed [11 and Figs 5,6].In fact, not only one exclusive geometrical method was used, but also different techniques to confine the beam/IMRT-beam lets within the optimal dose delivery zone. The initial approximations for conformal wedge filters involve promising approaches to be carried out.

The future applications of this mathematical framework show a number of alternatives. Among them, the industrial manufacturing/design of other types of BMD. The engineering precision for wedges/MLCs/LINACS, or BMD combinations. The improvement of the therapy treatment and the reduction of manufacturing economic cost. Besides, these equations constitute to enhance the design of the prospective conformal wedges.

To summarize, we presented for sharp learning in this contribution, a series of geometrical-mathematical formulation with both medical and industrial applications. We have set future perspectives for the new design of radiotherapy BMD/apparatus and other new types of BMD, such as the conformal wedge, to be developed in future.

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ANNEX A. BASIC GEOMETRIC PICS/SKETCHES

Figures 1 (upper) and 2. Geodesic calculations method. Upper pic shows the starting point to trace the curve. Pic 2 the final point at the plane quadrant.
Figures 3 (upper) and 4.- Geodesic calculations method, with cylinder intersection. Lower pic shows the Decomposition Approximation.
Fig 5.- This sketch [11] shows with more detail the beam decomposition that was used in previous contributions to determine the exact/approximated path of the pencil beam through the wedge.

Fig 6.- 2D approximation for wedge filters by Ulmer and Harder [35]. This approximation has acceptable experimental results that corroborate the formula [21]. But note that if we treat this problem in 2D, there is always an error depending on coordinate v (through the plane of the image).
Table 1.- Example of table with simulations of distance through standard wedge presented in previous publications [11].

<table>
<thead>
<tr>
<th>SIMULATIONS RESULTS FOR 30° WEDGE (Standard Manufacturing Size)</th>
</tr>
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<tbody>
<tr>
<td>ANGLE=30°</td>
</tr>
<tr>
<td>SIMULATION</td>
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<td>2</td>
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