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Sufficient conditions for subclasses of Certain meromorphic functions

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***Abstract*— The object of the present paper is to consider some sufficient conditions for certain subclasses of meromorphic functions in the punctured unit disc.**

***Keywords*—Meromorphic close-to-convex function, Star like function, Convex function.**

I. INTRODUCTION

Let Σ be the class of functions f of the form

$$f(z)=\frac{1}{z}+\sum_{n=1}^{\infty}a_nz^n \qquad (1)$$

which are analytic in the punctured unit disk $D=\{z\in C:0<|z|<1\}$. A function f belonging to the class Σ is said to be meromorphic starlike of order α if it satisfies

$$-Re\{\frac{zf'(z)}{f(z)}\}>\alpha \quad (z\in D) \qquad (2)$$

for some real $\alpha(0\le\alpha<1)$ .We denote this class by $\Sigma^*(\alpha)$.

A function $f(z)\in\Sigma$ is called meromorphic convex of order α if and only if

$$-Re\{1+\frac{zf''(z)}{f'(z)}\}>\alpha \quad (z\in D) \qquad (3)$$

for some $\alpha(0\le\alpha<1)$ .We denote it by $MK(\alpha)$.

A function $f(z)\in\Sigma$ is called meromorphic close-to-convex of order α if it satisfies

$$-Re\{z^2f'(z)\}>\alpha \quad (z\in D). \qquad (4)$$

We denote it by $MC(\alpha)$.

Several authors [1,2,3,4,5] have studied various subclasses of $\Sigma^*(\alpha)$, as well as subclasses of meromorphic convex functions of order α. Also several authors [6,7,8] have studied meromorphic close-to-convex of order α . We shall unify these functions in Definition 1.1.

Definition 1.1. Let $g(z)$ be a meromorphic starlike function in Σ, Let $MC^*(\alpha)$ be the class of functions $f(z)\in\Sigma$ satisfying the following inequality

$$-Re\{\frac{zf'(z)}{g(z)}\}>\alpha, z\in D \qquad (5)$$

for some $\alpha(0\le\alpha<1)$. The function f can also be called a meromorphic close-to-convex function.

II. THE MAIN RESULTS

In order to establish our main results,we require the following lemma:

Lemma 2.1. (see [9]) Let $p(z)=1+\sum_{n=k\ge1}^{\infty}a_nz^n$ be analytic in the unit disc U and $\alpha(0<\alpha\le1/2)$ be a positive real number. Then suppose that there exists a point $z_0\in U$ such that

$$Re\{p(z)\}>\alpha \; for \; |z|<|z_0| \qquad (6)$$

and

$$Re\{p(z_0)\}=p(z_0)=\alpha. \qquad (7)$$

Then we have

$$\frac{z_0p'(z_0)}{p(z_0)}\le -k(1-\alpha) \qquad (8)$$

Lemma 2.2. Let $p(z)=1+\sum_{n=1}^{\infty}a_nz^n$ be analytic in the unit disc U and $\alpha(0<\alpha\le1/2)$ be a positive real number. Suppose also that for arbitrary $r(0<r<1)$, such that

$$\min_{|z|\le r} Re\{p(z)\}=\min_{|z|\le r}|p(z)| \qquad (9)$$

and

$$Re\{\frac{zp'(z)}{p(z)}\}>\alpha-1, z\in U. \qquad (10)$$

Then we have

$$Re\{p(z)\}>\alpha, z\in U. \qquad (11)$$

Proof. Suppose that there exists a point $z_0\in U$ such that

$$Re\{p(z)\}>\alpha \; for \; |z|<|z_0| \qquad (12)$$

and

$$Re\{p(z_0)\}=\alpha, 0<\alpha\le\frac{1}{2}. \qquad (13)$$

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From the hypothesis of Lemma 2.2, then we have

$$Re\{p(z_0)\} = p(z_0) = \alpha, \quad 0 < \alpha \le \frac{1}{2}. \qquad (14)$$

From Lemma 2.1, then we have

$$Re\{\frac{z_0 p'(z_0)}{p(z_0)}\} \le \alpha - 1, \quad 0 < \alpha \le \frac{1}{2}. \qquad (15)$$

This contradicts the hypothesis (10) of Lemma 2.2 and it completes the proof of Lemma 2.2.

Theorem 2.1. Let $f(z) \in \Sigma$, and $\alpha(0 < \alpha \le 1/2)$ be a positive real number. Suppose that there exists a meromorphic star like function $g(z)$ such that

$$\min_{|z|\le r} Re\{-\frac{zf'(z)}{g(z)}\} = \min_{|z|\le r} |-\frac{zf'(z)}{g(z)}| \qquad (16)$$

for arbitrary $r(0 < r < 1)$, and

$$1 + Re\{\frac{zf''(z)}{f'(z)}\} < Re\{\frac{zg'(z)}{g(z)}\} - \alpha + 1, z \in D, \qquad (17)$$

where $0 < \alpha \le \frac{1}{2}$.Then we have $f(z) \in MC^*(\alpha)$.

Proof. Let

$$p(z) = -\frac{zf'(z)}{g(z)}, \qquad (18)$$

then $p(z)$ is analytic in U and $p(0) = 1$.Now using (18), we have

$$-1 - \frac{zf''(z)}{f'(z)} + \frac{zg'(z)}{g(z)} = \frac{zp'(z)}{p(z)}. \qquad (19)$$

Therefore, applying Lemma 2.2 and the hypothesis (16), (17) in Theorem 2.1, we have

$$-Re\{\frac{zf'(z)}{g(z)}\} > \alpha, \quad 0 < \alpha \le \frac{1}{2}, z \in D. \qquad (20)$$

This completes the proof of the theorem.

Corollary 2.1. Let $f(z) \in \Sigma$, and $\alpha(0 < \alpha \le 1/2)$ be a positive real number. Suppose that for arbitrary $r(0 < r < 1)$, $f(z)$ satisfies the condition

$$\min_{|z|\le r} Re\{z^2 f'(z)\} = \min_{|z|\le r} |z^2 f'(z)| \qquad (21)$$

and

$$Re\{\frac{zf''(z)}{f'(z)}\} < -1 - \alpha, 0 < \alpha \le \frac{1}{2}, z \in D. \qquad (22)$$

Then we have $f(z) \in MC(\alpha)$.

Proof. Let $g(z) = 1/z$ in Theorem 2.1,we can obtain Corollary 2.1.

Lemma 2.3. (see [9]) Let $p(z) = 1 + \sum_{n=k\ge1}^{\infty} a_n z^n$ be analytic in the unit disc U and $\alpha(1/2 < \alpha < 1)$ be a positive real number. Then suppose that there exists a point $z_0 \in U$ such that

$$Re\{p(z)\} > \alpha \; for \; |z| < |z_0| \qquad (23)$$

and

$$Re\{p(z_0)\} = p(z_0) = \alpha. \qquad (24)$$

Then we have

$$\frac{z_0 p'(z_0)}{p(z_0)} \le -\frac{k}{2}(2 - \alpha). \qquad (25)$$

Lemma 2.4. Let $p(z) = 1 + \sum_{n=1}^{\infty} a_n z^n$ be analytic in the unit disc U and $\alpha(1/2 < \alpha < 1)$ be a positive real number. Suppose also that for arbitrary $r(0 < r < 1)$, such that

$$\min_{|z|\le r} Re\{p(z)\} = \min_{|z|\le r} |p(z)| \qquad (26)$$

and

$$Re\{\frac{zp'(z)}{p(z)}\} > \frac{\alpha}{2} - 1, z \in U. \qquad (27)$$

Then we have

$$Re\{p(z)\} > \alpha, \quad z \in U. \qquad (28)$$

Proof. Suppose that there exists a point $z_0 \in U$ such that

$$Rep(z) > \alpha \quad for \quad |z| < |z_0| \qquad (29)$$

and

$$Re\{p(z_0)\} = \alpha, \quad \frac{1}{2} < \alpha < 1. \qquad (30)$$

By the hypothesis of Lemma 2.4, we have

$$Re\{p(z_0)\} = p(z_0) = \alpha, \quad \frac{1}{2} < \alpha < 1. \qquad (31)$$

Making use of Lemma 2.3, then we have

$$Re\{\frac{z_0 p'(z_0)}{p(z_0)}\} \le \frac{\alpha}{2} - 1, \frac{1}{2} < \alpha < 1. \qquad (32)$$

This contradicts the hypothesis (27) of Lemma 2.4 and it completes the proof of Lemma 2.4.

Theorem 2.2. Let $f(z) \in \Sigma$, and $\alpha(1/2 < \alpha < 1)$ be a positive real number. Suppose that there exists a meromorphic star like function $g(z)$ such that

$$\min_{|z|\le r} Re\{-\frac{zf'(z)}{g(z)}\} = \min_{|z|\le r} |-\frac{zf'(z)}{g(z)}| \qquad (33)$$

for arbitrary $r(0 < r < 1)$, and

$$1 + Re\{\frac{zf''(z)}{f'(z)}\} < Re\{\frac{zg'(z)}{g(z)}\} - \frac{\alpha}{2} + 1, z \in D. \qquad (34)$$

Then we have $f(z) \in MC^*(\alpha)$.

Proof. Let

$$p(z) = -\frac{zf'(z)}{g(z)}, \qquad (35)$$

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then $p(z)$ is analytic in U and $p(0)=1$. Now using (35), it follows that

$$-1-\frac{zf''(z)}{f'(z)}+\frac{zg'(z)}{g(z)}=\frac{zp'(z)}{p(z)}. \qquad (36)$$

By Lemma 2.4 and the hypothesis (33), (34) in Theorem 2.2, we obtain

$$-Re\{\frac{zf'(z)}{g(z)}\}>\alpha, \frac{1}{2}<\alpha<1, z\in D. \qquad (37)$$

This completes the proof of the Theorem 3.2.

Corollary 2.2. Let $f(z)\in\Sigma$, and $\alpha(1/2<\alpha<1)$ be a positive real number. Suppose that there for arbitrary $r(0<r<1)$

$$\min_{|z|\le r} Re\{z^2 f'(z)\}=\min_{|z|\le r} |z^2 f'(z)| \qquad (38)$$

and

$$Re\{\frac{zf''(z)}{f'(z)}\}<-1-\frac{\alpha}{2}, 1/2<\alpha<1, z\in D. \qquad (39)$$

Then we have $f(z)\in MC(\alpha)$.

Proof. Let $g(z)=1/z$ in Theorem 2.2, we can obtain Corollary 2.2.

REFERENCES

[1] J. Clunie, On meromorphic schlicht functions, J. London Math. Soc., vol. 34, pp. 215-216, 1959.

[2] N. E. Cho and S. Owa, Sufficient conditions for meromorphic starlikeness and close-to- convexity of order α, Int. J. Math. Math. Sci., vol. 26,no. 5 , pp. 317-319, 2001.

[3] M. Nunokawa and O. P. Ahuja, On meromorphic starlike and convex functions, Indian J. Pure Appl. Math., vol. 32, no. 7, pp. 1027-1032, 2001.

[4] B. A. Uralegaddi and A. R. Desai, Integrals of meromorphic starlike functions with positive and fixed second coefficients, J. Indian Acad. Math., vol. 24, no. 1 , pp. 27-36, 2002.

[5] R. M. Ali and V. Ravichandran, Classes of meromorphic α-convex functions, Taiwanese Journal Of Mathematics,vol. 14, no. 4, pp. 1479-1490, 2010.

[6] M. D. Ganigi and B.A.Uralegaddi, Subclasses of meromorphic close-to-convex functions, Bull. Math. Soc.Sci. Math.R. S. Rouma- nie (N.S.) , vol. 33,no. 81,no.2, pp. 105-109, 1989.

[7] N.E.Cho and S.Owa, Sufficient conditions for meromorphic starlikeness and close-to-convexity of order α, Journal of Mathematics and Mathematical Sciences , vol. 26, no. 5, pp. 317-319, 2001.

[8] B. A. Uralegaddi and A. R. Desai, On some criteria for close-to-convexity of meromorphic functions Journal of mathematical research Exposition, vol. 2, pp.301-311, 2000.

[9] J. Sokol and M. Nunokawa, On some sufficient conditions for univalence and starlikeness, Journal of Inequalities and Applications, vol. 2012, 2012.