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A Unique Common Fixed Point Theorem for Two pairs of Weakly Compatible Maps on Cone Metric Spaces

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Abstract- *In this paper we study for A Unique Common Fixed Point Theorem for Two pairs of weakly compatible Maps on Cone Metric spaces. Our result generalizes and improves some known recent results.*

Index Terms: **Weakly compatible maps, Common fixed point, Cone matric spaces**

I. INTRODUCTION

Huang and Zhang [3] recently generalize the concept of cone metric spaces and establish some fixed point theorems for contracting mapping. Later, several authors [1,4,5,6] have genereralized the results of Huang and Zhang [3]. In this paper, we prove a unique common fixed point theorem for two pairs of weakly compatible maps on cone metric spaces, which generalize and extends the results of Abbas and Rhoades [4] and others.

In this section we recall the definition of cone metric spaces and some of their properties. The following notions will be used in order to prove the main result.

Definitions1.1 Let E be a real Banach Space and P a subset of E. The set P is called a cone if and only if

(i) P is closed, non-empty and $P \neq \{0\}$

(ii) $0 \leq a, b \in R$ and $x, y \in P \Rightarrow ax + by \in P$.

(iii) $P \cap (-P) = \{0\}$

For a given cone $P \subseteq E$, we can define a partial ordering $\subseteq$ on E with respect to P by $x \leq y$ if and only if $y - x \in P$. We will write $x < y$ if $x \leq y$ and $x \neq y$, while $x << y$ will stands for $y - x \in$ Int. P, where Int .P denotes the interior of the set P.

Definitions1.2 Let X be a non-empty set. Suppose that the mapping $d : X \times X \to E$ satisfies.

(a) $0 \leq d(x, y)$, for all $x, y \in X$ and $d(x, y) = 0$ if and only if $x = y$;

(b) $d(x, y) = d(y, x)$ for all $x, y \in X$;

(c) $d(x, y) \leq d(x, z) + d(y, z)$ for all $x, y, z \in X$.

Then d is called a cone metric on X and (X, d) is called a cone metric space.

Definitions1.3 Let E be a Banach Space and $P \subset E$ a cone. The cone P is called normal if there is a number $K > 0$ such that for all $x, y \in E$,

$0 \leq x \leq y$ implies $\| x \| \leq K \| y \|$ .... (1)

The least positive number K satisfying the above inequality is called the normal constant of P. In the following we always suppose that E is a Banach space, P is a cone in E, which is partial ordering with respect to P.

Definitions1.4 Let (X,d) be a cone metric space. Let $\{x_n\}$be a sequence in X and $x \in X$.

(i) $\{x_n\}$ converges to x if for every $c \in E$ with $0 << c$, there is an n_0 such that for all $n \geq n_0$, $d(x_n, x) << c$. we denote this by $\lim_{n \to \infty}$ or $x_n \to x$, $(n \to \infty)$.

(ii) If for any $c \in E$ with $0 << c$, there is an n_0 such that for all $n,m \geq n_0$, $d(x_n, x_m) << c$ then $\{x_n\}$ is called a cauchy sequence in X.(X,d) is called a complete cone metric space, if every sequence in X is convergent in X.

Definitions1.5 Two self mapping f and g of a set X are said to be weakly compatible if they are commute at their concidence points, that is if fu = gu for some $u \in X$ then fgu = gfu.

Definitions1.6 Let f and g be self mappings of a set X . If w = fx =gx , for some $x \in X$, then x is called coincidence point of f and g, where w is called a point of coincidence of f and g.

II. MAIN RESULTS

K.P.R.Rao, Md. Mustaq Ali & N. Srinivasa Rao [2] have proved the following theorem:-

Theorem 2.1 Let (X, d) be a cone metric space and P be an order cone.Let f,g,S and T :$X \to X$ be such that

(i) $f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$ and one of T(X) and S(x) is complete subspace of X.

(ii) $d(fx,gx) \leq \lambda u$ where
$u \in [d(Sx, Ty), d(fx,Sx), d(gy,Ty), \frac{1}{2}\{d(fx,Ty)+d(gy,Sx)\}]$

for every $x,y \in X$ and for some $\lambda \in (0,1)$,

(iii) (f,S), and (g,T) are weakly compatible pairs.Then f, g, S and T have a unique
Common fixed point.

Now we prove the following theorem:-

Theorem 2.2 Let (X, d) be a cone metric space and P be an order cone.Let f,g,S and T :$X \to X$ be such that

(i) $f(X) \subseteq T(X)$, $g(X) \subseteq S(X)$ and one of T(X) and S(x) is complete subspace of X.

(ii) $d(Sx,Ty) \leq ad(fx, gy)+b[d(fx,Sx)+d(gy,Ty)] + \frac{1}{2}c[d(fx,Ty)+d(gy,Sx)]$ …(2)

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For all $x, y \in X$, where $a, b, c \geq 0$ and $2a + 2b + c < 1$. Suppose that the Pairs (f,S), and (g,T) are weakly compatible, then f, g, S and T have a unique Common fixed point.

Proof. Let $x_0 \in X$ .Define

$$y_{2n} = Sx_{2n} = gx_{2n+1}$$
$$y_{2n+1} = Tx_{2n+1} = fx_{2n+2}$$

Putting $x=x_{2n}$ and $y= x_{2n+1}$, for all n=1, 2, 3 …

By (2), we have

$$d(y_{2n}, y_{2n+1}) = d(Sx_{2n}, Tx_{2n+1}) \leq ad(fx_{2n}, gx_{2n+1}) + b[d(fx_{2n},Sx_{2n}) + d(gx_{2n+1}, Tx_{2n+1})] + \frac{1}{2}c[d(fx_{2n}, Tx_{2n+1}) + d(gx_{2n+1}, Sx_{2n})]$$
$$\leq ad(y_{2n-1}, y_{2n}) + b[d(y_{2n-1}, y_{2n}) + d(y_{2n}, y_{2n+1})] + \frac{1}{2}c[d(y_{2n-1}, y_{2n+1}) + d(y_{2n}, y_{2n+1})]$$
$$\leq (a + b + \frac{1}{2}c)d(y_{2n-1}, y_{2n}) + (b + \frac{1}{2}c)d(y_{2n}, y_{2n+1}).$$

Which implies that

$$d(y_{2n},y_{2n+1}) \leq \frac{a+b+\frac{1}{2}c}{1-(b+\frac{1}{2}c)} d(y_{2n-1},y_{2n})$$

Or, $d(y_{2n},y_{2n+1}) \leq \delta\, d(y_{2n-1},y_{2n})$

Where $\delta = \dfrac{a+b+\frac{1}{2}c}{1-(b+\frac{1}{2}c)} < 1$.

Similarly it can be shown that,

$$d(y_{2n+1},y_{2n+2}) \leq \delta\, d(y_{2n},y_{2n+1})$$

There, for all n,

$$d(y_{n+1},y_{n+2}) \leq d(y_n, y_{n+1}) \leq \ldots\ldots \leq (\delta)^{n-1} d(y_0,y_1).$$

Now, for any $m,n \to \infty$

$$d(y_n,y_m) \leq d(y_n, y_{n+1}) + d(y_{n+1},y_{n+2}) + \ldots\ldots\ldots + d(y_{m-1},y_m)$$
\
$$\leq [(\delta)^n + (\delta)^{n+1} + \ldots.. + (\delta)^{m-1}] d(y_1,y_0)$$
$$\leq \frac{\delta^n}{1-\delta^n} d(y_1,y_0).$$

From (1) , we have

$$||d(y_n,y_m)|| \leq \frac{\delta^n}{1-\delta^n} K ||d(y_1,y_o)||$$

which implies that $d(yn, ym) \to 0$ as $n,m \to \infty$. Hence {yn} is a Cauchy sequence. Since X is complete, there exists a point z in X such that $\lim_{n\to\infty}\{yn\}=z$, $\lim_{n\to\infty} Sx_{2n} = \lim_{n\to\infty} gx_{2n+1} = z$ and $\lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} fx_{2n+2} = z$ i.e.,

$$\lim_{n\to\infty} Sx2n = \lim_{n\to\infty} gx_{2n+1} = \lim_{n\to\infty} Tx_{2n+1} = \lim_{n\to\infty} fx_{2n+2} = z.$$

Since $T(X) \subseteq f(X)$, there exists a point $u \in X$ such that $z = fu$.

Then, by (2),we have,

$$d(Su, z) \leq d(Su,Tx_{2n-1}) + d(Tx_{2n-1}, z) \leq ad(fu, gx_{2n-1}) + b[d(fu, Su) + d(gx_{2n-1}, Tx_{2n-1})] + \frac{1}{2}c[d(fu, Tx_{2n-1}) + d(gx_{2n-1}, Su)] + d(Tx_{2n-1}, z)$$

By (1); we have

$$||d(Su, z)|| \leq aK ||d(fu, gx_{2n-1})|| + bK || [d(fu, Su) + d(gx_{2n-1}, Tx2_{n-1})] || + \frac{1}{2}cK || [d(fu, Tx_{2n-1}) + d(gx_{2n-1}, Su)] || + K ||d(Tx_{2n-1}, z)||$$

Taking the limit as $n \to \infty$ yields

$$d(Su, z) \leq ad(z, z) + b[d(z, Su) + d(z, z)] + \frac{1}{2}c[d(z, z) + d(z, Su)] + d(z, z)$$
$$\leq (b + \frac{1}{2}c)d(Su, z).$$

Which is a contradiction since $2a + 2b + c < 1$. Therefore $Su = fu = z$. Since $S(X) \subseteq g(X)$, there exists a point $v \in X$ such that $z = gv$. Then, by (2), we have

$$d(z, Tv) \leq d(Su, Tv) \leq ad(fu, gv) + b[d(fu, Su) + d(gv, Tv)] + \frac{1}{2}c[d(fu, Tv) + d(gv, Su)]$$

$$\leq ad(z, z) + b[d(z, z) + d(z, Tv)] + \frac{1}{2}c[d(z, Tv) + d(z, z)]$$

$$\leq (b + \frac{1}{2}c)d(z, Tv)$$

which is a contradiction since 2a + 2b + c < 1. Therefore Tv = gv = z.
Thus Su = fu = Tv = gv = z.

Since f and S are weakly compatible maps, then Sfu = fSu i.e., Sz = fz. Now we show that z is a fixed point of S. if Sz $\neq$ z, then by (2), we have

$$\begin{aligned} d(Sz, z) &\leq d(Sz, Tv) \\ &\leq ad(fz, gv) + b[d(fz, Sz) + d(gv, Tv)] \\ &\quad + \frac{1}{2}c[d(fz, Tv) + d(gv, Sz)] \\ &\leq ad(Sz, z) + b[d(Sz, Sz) + d(z, z)] \\ &\quad + \frac{1}{2}c[d(Sz, z) + d(z, Sz)] \\ &\leq (a + c)d(Sz, z) \end{aligned}$$

Which is a contradiction since 2a + 2b + c < 1. Therefore Sz = z. Hence Sz = fz = z.
Similarly, g and T are weakly compatible maps, we have Tz = gz. Now we show that z is a fixed point of T.
If Tz $\neq$ z, then by (2), we have

$$\begin{aligned} d(z, Tz) &\leq d(Sz, Tz) \\ &\leq ad(fz, gz) + b[d(fz, Sz) + d(gz, Tz)] \\ &\quad + \frac{1}{2}c[d(fz, Tz) + d(gz, Sz)] \\ &\leq ad(z, Tz) + b[d(z, z) + d(Tz, Tz)] \\ &\quad + \frac{1}{2}c[d(z, Tz) + d(Tz, z)] \\ &\leq (a + c)d(z, Tz) \end{aligned}$$

which is a contradiction since 2a + 2b + c < 1. Therefore Tz = z. Hence Tz = gz = z.
Thus Sz = Tz = fz = gz = z, i.e. z is a common fixed point of f, g, S and T.

Finally, in order to prove the uniqueness of z, suppose w is another common fixed points of f, g, S and T respectively. then by (2), we have

$$\begin{aligned} d(z,w) &\leq d(Sz,Tw) \\ &\leq ad(fz, gw) + b[d(fz, Sz) + d(gw, Tw)] \\ &\quad + \frac{1}{2}c[d(fz, Tw) + d(gw, Sz)] \\ &\leq ad(z,w) + b[d(z, z) + d(w,w)] \\ &\quad + \frac{1}{2}c[d(z,w) + d(w, z)] \\ &\leq (a + c)d(z,w) \end{aligned}$$

Which is a contradiction since 2a + 2b + c < 1.
Therefore z = w.
Hence z is the unique common fixed point of f, g, S and T respectively.

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