Thermoelastic Analysis of a Hollow Cylinder with Radiation Conditions

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II. STATEMENT OF THE PROBLEM

Consider a hollow cylinder of length 2h in which sources are generated according to linear function of temperature. The material is isotropic, homogeneous and all properties are assumed to be constant. The equation for heat conduction is [8]:

\[ k \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial \theta}{\partial r} \right) + \frac{\partial^2 \theta}{\partial z^2} \right] + \Theta(r, z, t, \theta) = \frac{\partial \theta}{\partial t} \]  (1)

where \( \Theta(r, z, t, \theta) \) is the source function and \( k = \frac{\lambda}{\rho C} \), \( \lambda \) being the thermal conductivity of the material, \( \rho \) is the density and \( C \) is the calorific capacity, assumed to be constant.

For convenience, we consider the under given functions as the superposition of the simpler function [9]:

\[ \Theta(r, z, t, \theta) = Q(r, z, t) + \psi(t) \theta(r, z, t) \]  (2)

and

\[ T(r, z, t) = \Theta(r, z, t) \exp \left[ - \int_0^t \psi(\zeta) d\zeta \right] \]  (3)

\[ \chi(r, z, t) = \phi(r, z, t) \exp \left[ - \int_0^t \psi(\zeta) d\zeta \right] \]

For the sake of convenience, consider

\[ \chi(r, z, t) = \frac{\delta(r - r_0)}{2\pi r_0} v(z) u(t) \]  (4)

where \( v(z) \), \( u(t) \) are arbitrary functions.

Substituting equations (2) and (3) in the heat conduction equation (1), one obtain

\[ k \left[ \frac{1}{r} \frac{\partial}{\partial r} \left( r \frac{\partial T}{\partial r} \right) + \frac{\partial^2 T}{\partial z^2} \right] + \chi(r, z, t) = \frac{\partial T}{\partial t} \]  (5)

where \( k \) is the thermal diffusivity of the material of the cylinder (which is assumed to be constant). Subject to the initial and boundary condition...
\[ M_r(T, 1, 0, 0) = 0 \]  
(6)

\[ M_r(T, 1, k_1, a) = 0 \text{ for all } -h \leq z \leq h, \ t > 0 \]  
(7)

\[ M_c(T, 1, k_2, b) = 0 \text{ for all } -h \leq z \leq h, \ t > 0 \]  
(8)

\[ M_z(T, 1, k_3, h) = u(t) \delta(r-r_0) \]  
(9)

\[ M_z(T, 1, k_4, -h) = 0, \text{ for all } a \leq r \leq b, \ t > 0 \]  
(10)

The most general expression for these conditions can be given by

\[ M_v(f, k, \delta, s) = (k^2 + \delta f)_{v=4} \]

where the prime (') denotes differentiation with respect to \( v \); \( \delta(r-r_0) \) are the Dirac Delta functions having \( a \leq r_0 \leq b \); \( u(t) \delta(r-r_0) \) is the additional sectional section.

\[ \text{heat available on its surface at } z = h \text{ and } k, \delta \text{ are radiation constants on the upper and lower surface of cylinder respectively.} \]

The radiation and axial displacement \( U \) and \( W \) satisfy the uncoupled thermoelastic equation as [8] are

\[ \nabla^2 U - \frac{U}{r^2} + (1 - 2v)^{-1} \frac{\partial e}{\partial r} = 2 \left( \frac{1 + v}{1 - 2v} \right) \frac{\partial \theta}{\partial z} \]  
(11)

\[ \nabla^2 W + (1 + 2v)^{-1} \frac{\partial e}{\partial z} = 2 \left( \frac{1 + v}{1 - 2v} \right) \alpha_t \frac{\partial \theta}{\partial r} \]  
(12)

where

\[ e = \frac{\partial U}{\partial r} + \frac{U}{r} + \frac{\partial W}{\partial z} \]

is the volume dilatation,

\[ U = \frac{\partial \phi}{\partial r} \]  
(13)

\[ W = \frac{\partial \phi}{\partial z} \]  
(14)

The thermoelastic displacement function \( \phi(r, z, t) \) as [4] is governed by the Poisson’s equation

\[ \nabla^2 \phi = \left( \frac{1 + v}{1 - 2v} \right) \alpha_t \theta \]  
(15)

with \( \phi = 0 \) at \( r = a \) and \( r = b \).

\[ \nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}, \]

where \( v \) and \( \alpha_t \) are poisons ratio and the linear coefficient of thermal expansion of the material of the cylinder respectively.

The stress functions are given by [4] as

\[ T_c(a, z, t) = 0 \quad T_c(b, z, t) = 0 \quad T_c(r, 0, t) = 0 \]  
(17)

\[ \sigma_r(a, z, t) = p_1, \quad \sigma_r(b, z, t) = -p_0, \quad \sigma_z(r, 0, t) = 0 \]  
(18)

where \( p_1 \) and \( p_0 \) are the surface pressure assumed to be uniform over the boundaries of the cylinder. The stress functions are expressed in terms of displacement components by the relations [4]:

\[ \sigma_r = (\lambda + 2G) \frac{\partial U}{\partial r} + \lambda \left( \frac{U}{r} + \frac{\partial W}{\partial z} \right) \]  
(19)

\[ \sigma_z = (\lambda + 2G) \frac{\partial W}{\partial z} + \lambda \left( \frac{\partial U}{\partial r} + \frac{U}{r} \right) \]  
(20)

\[ \sigma_\theta = (\lambda + 2G) \frac{U}{r} + \lambda \left( \frac{\partial W}{\partial z} + \frac{\partial U}{\partial r} \right) \]  
(21)

\[ \tau_{rz} = G \left( \frac{\partial W}{\partial r} + \frac{\partial U}{\partial z} \right) \]  
(22)

where \( \lambda = \frac{2Gv}{1 - 2v} \) is the Lame’s constant, \( G \) is the shear modulus and \( U \) and \( W \) are the displacement components.

The equations (1) to (22) constitute the mathematical formulation of the problem under consideration.

### III. SOLUTION OF THE PROBLEM

In order to solve fundamental differential equation (5) under the boundary conditions (7) and (8), we first introduce the integral transform of order \( n \) over the variable \( r \). Let \( n \) be the parameter of the transform, then the integral transform and its inversion theorem is written as [1]

\[ \tilde{g}(r) = \int_a^b g(r) S_p(k_1, k_2, \mu_n r) \, dr, \]  
(23)

\[ g(r) = \sum_{n=1}^{\infty} \frac{\tilde{g}(n)}{C_n} S_p(k_1, k_2, \mu_n r) \]  
(24)
where \( \bar{g}_n(n) \) is the transformation of \( g(r) \) with respect to nucleus \( S_p(k_1, k_2, \mu, r) \).

Applying the transformation defined in equation (23) to the equations (5) (6) and (9) and using equation (7) and (8) one obtains

\[
k \left[ -\mu_n^2 \bar{T}(n, z, t) + \frac{\partial^2 \bar{T}(n, z, t)}{\partial z^2} \right] + \frac{v(z)}{2\pi} = u(t) S_0(k_1, k_2, \mu_n r_0) = \frac{\partial \bar{T}(n, z, t)}{\partial t} \tag{25}
\]

Where

\[
M_z(\bar{T}, 1, 0, 0) = 0 \tag{26}
\]

\[
M_z(\bar{T}, 1, k_3, h) = u(t) r_0 S_0(k_1, k_2, \mu_n r_0) \tag{27}
\]

\[
M_z(\bar{T}, 1, k_4, -h) = 0 \tag{28}
\]

Where \( \bar{T} \) is the transformed function of \( T \) and \( n \) is the transformed parameter. The eigen values \( \mu_n \) are the positive roots of the characteristic equation

\[
J_0(k_1, \mu a) Y_0(k_2, \mu b) - J_0(k_2, \mu b) Y_0(k_1, \mu a) = 0
\]

The kernel function \( S_0(k_1, k_2, \mu_n r) \) can be defined as

\[
S_0(k_1, k_2, \mu_n r) = J_0(\mu_n r)Y_0(k_1, \mu a) + Y_0(k_2, \mu a)
\]

\[
+ Y_0(\mu_n r) \left[ J_0(k_1, \mu a) + J_0(k_2, \mu b) \right]
\]

with

\[
J_0(k_1, \mu_n r) = J_0(\mu_n r) + k_1 \mu_n J'_0(\mu_n r)
\]

\[
Y_0(k_1, \mu_n r) = Y_0(\mu_n r) + k_1 \mu_n Y'_0(\mu_n r)
\]

for \( i = 1, 2 \) and

\[
C_n = \int_a^b [S_0(k_1, k_2, \mu b)]^2 dr
\]

in which

\[
J_0(\mu_n r) \quad \text{and} \quad Y_0(\mu_n r) \quad \text{are Bessel’s functions of the first and second kind of order} \quad p = 0 \quad \text{respectively. We introduce another integral transformation that respond to the radiation type boundary conditions as} \quad [3]:
\]

\[
\tilde{f}(m, t) = \int_{-h}^h f(z, t) P_m(z) \, dz
\]

Further applying the transform defined in equation (29) to the equation (25) and using (27) and (28), one obtains

\[
k \left[ -\mu_n^2 \bar{T}^*(n, m, t) + \frac{\partial^2 \bar{T}^*(n, m, t)}{\partial z^2} \right] + \frac{v(z)}{2\pi} = u(t) r_0 S_0(k_1, k_2, \mu_n r_0)
\]

\[
+ \frac{u(t)}{2\pi} S_0(k_1, k_2, \mu_n r_0) P_m(z) v(z) = \frac{d\bar{T}^*(n, m, t)}{dt} \tag{31}
\]

\[
M_z(\bar{T}^*, 1, 0, 0) = 0 \tag{32}
\]

where \( \bar{T}^* \) is the transformed function of \( T \) and \( m \) is the transformed parameter. The symbol (*) means a function in the transform domain and the nucleus is given by the orthogonal functions in the interval \(-h \leq z \leq h\) as

\[
P_m(z) = Q_m \cos(\xi_m z) - W_m \sin(\xi_m z)
\]

In which

\[
Q_m = \xi_m (k_3 + k_4) \cos(\xi_m h)
\]

\[
W_m = 2\cos(\xi_m h) + (k_3 - k_4) \xi_m \sin(\xi_m h)
\]

\[
\lambda_m = \int_{-h}^h p_m^2(z) \, dz = h[Q_m^2 + W_m^2] + \sin \left( \frac{2\xi_m h}{2\xi_m} \right) \left[ Q_m^2 - W_m^2 \right]
\]

The eigen values \( \xi_m \) are the positive roots of the characteristic equation

\[
[k_3a \cos(ah) + \sin(ah)] [\cos(ah) + k_4a \sin(ah)] = [k_3a \cos(ah) - \sin(ah)] [\cos(ah) - k_4a \sin(ah)]
\]

After performing some calculations on the equation (31), the reduction is made to linear first order differential equation as

\[
\frac{d\bar{T}^*}{dt} + k(\mu_n^2 + \xi_m^2) \bar{T}^* = \frac{p_m(h) - P_m(-h)}{k_4} \left[ r_0 + P_m(z) v(z) \right] \tag{33}
\]

The transformed temperature solution is

\[
\bar{T}^*(n, m, t) = \frac{Q(m, n)}{k(\xi_m^2 + \mu_n^2)} [u(t) - \exp(-k(\mu_n^2 + \xi_m^2)t)]
\]

where
\[
\Omega(m,n) = \left[ \frac{P_m(h)}{k_3} - \frac{P_m(-h)}{k_4} \right] k_6 + \frac{P_0(z)(y)}{2\pi} S_0(k_1, k_2, \mu_n, r_0)
\]

(35)

Applying the inversion of transformation rules defined in equations (24) and (30) the temperature solution is shown as follows

\[
T(r, z, t) = \sum_{n=1}^{\infty} \frac{1}{C_n} \sum_{m=1}^{\infty} \frac{\Omega(\xi_m, \mu_n)}{\lambda_m k(\xi_m^2 + \mu_n^2)} \times [u(t) - \exp(-k(\xi_m^2 + \mu_n^2)t)] P_m(z)
\]

\[
\times S_0(k_1, k_2, \mu_n, r)
\]

(36)

Taking into account of the first equation of equation (3), the temperature distribution is finally represented by

\[
\theta(r, z, t) = \sum_{n=1}^{\infty} \frac{1}{C_n} \sum_{m=1}^{\infty} \frac{\Omega(\xi_m, \mu_n)}{\lambda_m k(\xi_m^2 + \mu_n^2)} \times [u(t) - \exp(-k(\xi_m^2 + \mu_n^2)t)] P_m(z)
\]

\[
\times S_0(k_1, k_2, \mu_n, r) \exp\left[\int_0^t \psi(\zeta) d\zeta\right]
\]

(37)

The equation (37) represents the required temperature distribution at any instant and at all points of the hollow cylinder when there are conditions of radiation type.

**IV. DETERMINATION OF THERMOELASTIC DISPLACEMENT**

Substituting value of temperature distribution \(\theta(r, z, t)\) from equation (37) in equation (15) one obtains the thermoelastic displacement function \(\phi(r, z, t)\) as

\[
\phi(r, z, t) = \left[1 + \frac{v}{1 - v}\right] \alpha \sum_{n=1}^{\infty} \frac{1}{C_n} \sum_{m=1}^{\infty} \frac{\Omega(\xi_m, \mu_n)}{\lambda_m k(\xi_m^2 + \mu_n^2)} \times [u(t) - \exp(-k(\xi_m^2 + \mu_n^2)t)] P_m(z)
\]

\[
\times S_0(k_1, k_2, \mu_n, r) \exp\left[\int_0^t \psi(\zeta) d\zeta\right]
\]

(38)

Substituting the value of \(\phi(r, z, t)\) from equation (38) in equations (13) and (14) one obtains

\[
U = \left[1 + \frac{v}{1 - v}\right] \alpha \sum_{n=1}^{\infty} \frac{1}{C_n} \sum_{m=1}^{\infty} \frac{\Omega(\xi_m, \mu_n)}{\lambda_m k(\xi_m^2 + \mu_n^2)} \times [u(t) - \exp(-k(\xi_m^2 + \mu_n^2)t)] P_m(z)
\]

\[
\times S_0(k_1, k_2, \mu_n, r) \exp\left[\int_0^t \psi(\zeta) d\zeta\right]
\]

(41)

The equation (37) represents the required temperature distribution at any instant and at all points of the hollow cylinder when there are conditions of radiation type.

**V. DETERMINATION OF STRESS FUNCTIONS**

The stress components can be evaluated by substituting the values of thermoelastic displacement from equations (39) and (40) in equations (19) to (22) one obtains

\[
\sigma_r = \left[1 + \frac{v}{1 - v}\right] \alpha \sum_{n=1}^{\infty} \frac{1}{C_n} \sum_{m=1}^{\infty} \frac{\Omega(\xi_m, \mu_n)}{\lambda_m k(\xi_m^2 + \mu_n^2)} \times [u(t) - \exp(-k(\xi_m^2 + \mu_n^2)t)] P_m(z)
\]

\[
\times S_0(k_1, k_2, \mu_n, r) \exp\left[\int_0^t \psi(\zeta) d\zeta\right]
\]

(42)
\[ \sigma_z = -\left( \frac{1 + v}{1 - v} \right) \alpha_t \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\Omega(\xi_m, \mu_n)}{C_n C_m (\xi_m^2 + \mu_n^2)^2} \frac{\lambda_m k}{\xi_m^2 + \mu_n^2} \]

\[ [u(t) - \exp(-k(\xi_m^2 + \mu_n^2)t)] P_m(z) \]

\[ \times \left[ \left( -\lambda + 2G \right) \frac{\xi_m^2}{\xi_m^2 + \mu_n^2} S_0(k_1, k_2, \mu_n r) \right. \]

\[ + \left. \lambda \left( \mu_n^2 S_0'(k_1, k_2, \mu_n r) + \frac{\mu_n S_0'(k_1, k_2, \mu_n r)}{r} \right) \right] \]

\[ \times \exp\left[ \int_0^t \psi(\xi) d\xi \right] \quad (43) \]

\[ \sigma_y = -\left( \frac{1 + v}{1 - v} \right) \alpha_t \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\Omega(\xi_m, \mu_n)}{C_n C_m (\xi_m^2 + \mu_n^2)^2} \frac{\lambda_m k}{\xi_m^2 + \mu_n^2} \]

\[ [u(t) - \exp(-k(\xi_m^2 + \mu_n^2)t)] P_m(z) \]

\[ \times \left[ (\lambda + 2G) \frac{\xi_m^2}{\xi_m^2 + \mu_n^2} S_0(k_1, k_2, \mu_n r) \right. \]

\[ + \left. \lambda \left( \mu_n^2 S_0'(k_1, k_2, \mu_n r) + \frac{\mu_n S_0'(k_1, k_2, \mu_n r)}{r} \right) \right] \]

\[ \times \exp\left[ \int_0^t \psi(\xi) d\xi \right] \quad (44) \]

\[ \tau_{rz} = -2G \left( \frac{1 + v}{1 - v} \right) \alpha_t \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\Omega(\xi_m, \mu_n)}{C_n (\xi_m^2 + \mu_n^2)^2} \frac{\lambda_m k}{\xi_m^2 + \mu_n^2} \]

\[ [u(t) - \exp(-k(\xi_m^2 + \mu_n^2)t)] \]

\[ \times \left[ (\frac{1}{\xi_m^2}) \left( Q_m \sin(\xi_m z) + W_m \cos(\xi_m z) \right) \right. \]

\[ \times \left. \mu_n S_0'(k_1, k_2, \mu_n r) \right] \]

\[ \times \exp\left[ \int_0^t \psi(\xi) d\xi \right] \quad (45) \]

VI. SPECIAL CASE AND NUMERICAL RESULTS

Set \( \psi(\zeta) = -\zeta \), \( v(z) = \delta(z - z_0) \).

\( u(t) = \exp(-\alpha t) \) \quad (46)

Using equation (46) in equations (36) to (45) one obtain the expression for temperature distribution, displacement and stresses respectively as follows:

\[ \theta(r, z, t) = \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\Omega(\xi_m, \mu_n)}{C_n C_m (\xi_m^2 + \mu_n^2)^2} \frac{\lambda_m k}{\xi_m^2 + \mu_n^2} \frac{1}{\xi_m^2 + \mu_n^2} - \omega \]

\[ \times \left[ \exp(-\alpha t) - \exp(-k(\xi_m^2 + \mu_n^2)t) \right] P_m(z) \]

\[ \times \exp(-t^2/2) \]

\[ \phi(r, z, t) = -\left( \frac{1 + v}{1 - v} \right) \alpha_t \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\Omega(\xi_m, \mu_n)}{C_n C_m (\xi_m^2 + \mu_n^2)^2} \frac{\lambda_m k}{\xi_m^2 + \mu_n^2} \frac{1}{\xi_m^2 + \mu_n^2} \]

\[ \times \left[ \exp(-\alpha t) - \exp(-k(\xi_m^2 + \mu_n^2)t) \right] P_m(z) \]

\[ \times \exp(-t^2/2) \]

\[ U(r, z, t) = -\left( \frac{1 + v}{1 - v} \right) \alpha_t \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\Omega(\xi_m, \mu_n)}{C_n C_m (\xi_m^2 + \mu_n^2)^2} \frac{\lambda_m k}{\xi_m^2 + \mu_n^2} \frac{1}{\xi_m^2 + \mu_n^2} \]

\[ \times \left[ \exp(-\alpha t) - \exp(-k(\xi_m^2 + \mu_n^2)t) \right] P_m(z) \]

\[ \times \exp(-t^2/2) \]

\[ W(r, z, t) = -\left( \frac{1 + v}{1 - v} \right) \alpha_t \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\Omega(\xi_m, \mu_n)}{C_n C_m (\xi_m^2 + \mu_n^2)^2} \frac{\lambda_m k}{\xi_m^2 + \mu_n^2} \frac{1}{\xi_m^2 + \mu_n^2} \]

\[ \times \left[ \exp(-\alpha t) - \exp(-k(\xi_m^2 + \mu_n^2)t) \right] \]

\[ \times \left( -\frac{1}{\xi_m^2} \right) \left( Q_m \sin(\xi_m z) + W_m \cos(\xi_m z) \right) \]

\[ \times \mu_n S_0'(k_1, k_2, \mu_n r) \exp(-t^2/2) \]

\[ \sigma_r = -\left( \frac{1 + v}{1 - v} \right) \alpha_t \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\Omega(\xi_m, \mu_n)}{C_n C_m (\xi_m^2 + \mu_n^2)^2} \frac{\lambda_m k}{\xi_m^2 + \mu_n^2} \frac{1}{\xi_m^2 + \mu_n^2} \]

\[ \times \left[ \exp(-\alpha t) - \exp(-k(\xi_m^2 + \mu_n^2)t) \right] \]

\[ \times \left( -\frac{1}{\xi_m^2} \right) \left( Q_m \sin(\xi_m z) + W_m \cos(\xi_m z) \right) \]

\[ \times \mu_n S_0'(k_1, k_2, \mu_n r) \exp(-t^2/2) \]

\[ \times \exp(-t^2/2) \]

\[ \lambda \left( \mu_n^2 S_0'(k_1, k_2, \mu_n r) + \frac{\mu_n S_0'(k_1, k_2, \mu_n r)}{r} \right) \]

\[ \times \exp(-t^2/2) \]

\[ (\lambda + 2G) \mu_n^2 S_n''(k_1, k_2, \mu_n r) \]

\[ + \left( \mu_n^2 S_0'(k_1, k_2, \mu_n r) - \xi_m^2 S_0'(k_1, k_2, \mu_n r) \right) \]

\[ \times \exp(-t^2/2) \]

\[ \times \exp(-t^2/2) \]

\[ \times \exp(-t^2/2) \]
\[
\sigma_z = \left( \frac{1 + v}{1 - v} \right) \alpha_t
\]
\[
\times \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{\Omega(\xi_m, \mu_n)}{C_n} \left( \mu_n^2 + \xi_m^2 \right) \lambda_m \left( \left( \mu_n^2 + \xi_m^2 \right) k - \omega \right)
\]
\[
\times \left[ \exp(-\omega t) - \exp(-k(\mu_n^2 + \xi_m^2)t) \right] P_m(z)
\]
\[
\times \left[ - (\lambda + 2G) \xi_m^2 S_0(k_1, k_2, \mu_n r) \right.
\]
\[
\times \left. + \lambda \left( \mu_n^2 S_0(k_1, k_2, \mu_n r) + \mu_n^2 S_0'(k_1, k_2, \mu_n r) \right) \right]
\]
\[
\times \exp(-t^2 / 2)
\]
Fig.(1). Graph of temperature distribution

Fig.(2). Graph of displacement component

Fig.(3). Graph of thermoelastic displacement function

Fig.(4). Graph of radial stress distribution

Fig.(5). Graph of axial stress distribution

Fig.(6). Graph of tangential stress distribution
Dr. N.W. Khobragade for being M.Sc in Statistics and Maths he attained Ph.D. He has been teaching since 1986 for 27 years at PGTD of Maths, RTM Nagpur University, Nagpur and successfully handled different capacities. At present he is working as Professor. Achieved excellent experiences in Research for 15 years in the area of Boundary value problems and its application. Published more than 180 research papers in reputed journals. Fourteen students awarded Ph.D Degree and four students submitted their thesis in University for award of Ph.D Degree under their guidance.