

# Inverse Transient Thermoelastic Problem of a Flat Thin Square Plate Producing Deflection

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*Abstract- This paper deals with inverse transient thermoelastic problem of a flat square plate of uniform thickness with prescribed boundary conditions.. We determined unknown temperature distribution, displacement and stress functions of a plate. Numerical estimates for heating processes subjected to known heat supply on the boundary  $z = h$  have been obtained and depicted graphically.*

**Key words:** Transient response, square plate, temperature distribution, thermal stress, integral transform, inverse problem.

## I. INTRODUCTION

The inverse thermo elastic problem consists of the determination of the temperature of the heating medium and the heat flux of a solid when the conditions of the displacement and stresses are known at some points of the solid under consideration. This inverse problem is relevant to different industries where machinery such as the main shaft of lathe and turbine and roll of a rolling mill is subject to heating.

In [1], [4], [6] and [7], direct transient thermo elastic problems are considered and the heating temperature and the heat flux on the surface of an isotropic rectangular plate are derived. The inverse problems of thermo elasticity of a thin rectangular plate and annular disc are considered in [2] and [3].

In the present problem, an attempt is made to study the theoretical solution for an inverse transient thermo elastic problem to determine the unknown temperature, displacement and stress functions of the plate occupying the region with  $a$  as the length of sides and  $h$  as the small thickness with known interior heat flux. The thermal deflection of square plates with clamped edges has received considerable attention because of its technical importance. Finite integral transform techniques are used to find the solution of the problem. Numerical estimate for the temperature distribution on the boundary  $z = h$  is obtained.

## II. FORMULATION OF THE PROBLEM: GOVERNING EQUATION

Consider a flat square plate of thickness  $h$  occupying the space  $D = \{(x, y, z) \in R^3 : 0 \leq x \leq a, 0 \leq y \leq a, 0 \leq z \leq h\}$ . The displacement components  $u_x$ ,  $u_y$  and

$u_z$  in the  $x$ ,  $y$  and  $z$  directions, respectively are in the integral form as [2]

$$u_x = \int_0^a \left[ \frac{1}{E} \left( \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} - \nu \frac{\partial^2 U}{\partial x^2} \right) + \alpha T \right] dx \quad (1)$$

$$u_y = \int_0^a \left[ \frac{1}{E} \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial z^2} - \nu \frac{\partial^2 U}{\partial y^2} \right) + \alpha T \right] dy \quad (2)$$

$$u_z = \int_0^c \left[ \frac{1}{E} \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} - \nu \frac{\partial^2 U}{\partial z^2} \right) + \alpha T \right] dz \quad (3)$$

where  $E$ ,  $\nu$  and  $\alpha$  are the Young's modulus, Poisson's ratio and the linear coefficient of thermal expansion of the material of the plate respectively and  $U(x, y, z, t)$  is the Airy's stress functions which satisfy the differential equation [2]:

$$\left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right)^2 U(x, y, z, t) = -\alpha E \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2} \right) T(x, y, z, t) \quad (4)$$

where  $T(x, y, z, t)$  denotes the temperature of a flat square plate satisfying the following differential equation [2]

$$\frac{\partial^2 T}{\partial x^2} + \frac{\partial^2 T}{\partial y^2} + \frac{\partial^2 T}{\partial z^2} = \frac{1}{\kappa} \frac{\partial T}{\partial t} \quad (5)$$

where  $\kappa$  is the thermal diffusivity of the substance, subject to the initial conditions

$$T(x, y, z, t) \Big|_{t=0} = 0 \quad (6)$$

the boundary conditions

$$T(x, y, z, t) \Big|_{x,y=0,a} = 0 \quad (7)$$

$$T(x, y, z, t) \Big|_{z=0} = u(x, y, t) \quad (8)$$

$$T(x, y, z, t) \Big|_{z=h} = g(x, y, t) \quad (9)$$

the interior condition

$$T(x, y, z, t) \Big|_{z=\xi} = f(x, y, t) \quad (10)$$

where the function  $u(x, y, t)$  and  $f(x, y, t)$  is assumed to be known while the function  $g(x, y, t)$  is not.

The stress components in terms of  $U(x, y, z, t)$  are given by

$$\sigma_{xx} = \sigma_{yy} = \left( \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} \right) \quad (11)$$

$$\sigma_{zz} = \left( \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} \right) \quad (12)$$

The equations (1) to (12) constitute the mathematical formulation of the problem under consideration.

### III. SOLUTION OF THE HEAT CONDUCTION EQUATION

*Determination of the temperature  $T(x, y, z, t)$*

Applying finite Fourier sine transform twice as defined in [5] to the equations (5), (6), (9) and (10) over the variables  $x$  and  $y$  with respects to the boundary conditions (8 – 9) and taking the Laplace transform, one obtains

$$\bar{T}^*(m, n, z, s) = \bar{f}^*(m, n, s) \left[ \frac{\sinh w z}{\sinh w \xi} \right] - \bar{u}^*(m, n, s) \left[ \frac{\sinh w(z - \xi)}{\sinh w \xi} \right]$$

where  $w^2 = (m^2 + n^2)(\pi/a)^2 + s/\kappa$ ,

$\bar{T}^*(m, n, z, s)$  is the Laplace transformed function of  $\bar{T}(m, n, z, t)$ , and  $\bar{T}(m, n, z, t)$  is the Fourier sine transformed function of  $T(x, y, z, t)$ . Applying the inversion theorems of Fourier sine transform twice and taking inverse Laplace transform by means of complex contour integration and the residue theorem, one obtains the expression of the temperature distribution  $T(x, y, z, t)$  as

$$T(x, y, z, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \left( \frac{8\pi k l}{\xi^2} \right) \varphi_{nl} \sin(m\pi x) \sin(n\pi y) \quad (13)$$

On the other hand, for the expression of unknown temperature gradient can be obtained using equation (13) into boundary conditions (9) as

$$g(x, y, t) = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \left( \frac{8\pi k l}{\xi^2} \right) \psi_{nl} \sin(m\pi x) \sin(n\pi y) \quad (14)$$

where

$$\varphi_{nl} = \int_0^t \left\{ \frac{\bar{u}(m, n, t') \sin[l\pi(z - \xi)/\xi] - \bar{f}(m, n, t') \sin[l\pi z/\xi]}{\cos[l\pi]} \right\} \times e^{-k((m^2 + n^2)(\pi/a)^2 + (l\pi/\xi)^2)(t-t')} dt'$$

$$\psi_{nl} = \int_0^t \left\{ \frac{\bar{u}(m, n, t') \sin[l\pi(h - \xi)/\xi] - \bar{f}(m, n, t') \sin[l\pi h/\xi]}{\cos[l\pi]} \right\} \times e^{-k((m^2 + n^2)(\pi/a)^2 + (l\pi/\xi)^2)(t-t')} dt'$$

in which  $m, n$  is the Fourier sine transform parameter.

### Determination of Airy's stress and displacement functions

Substituting the value of (13) in (4) one obtains

$$U(x, y, z, t) = \frac{8\alpha E k}{\pi} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \left( \frac{l a^2}{(m^2 + n^2)\xi^2 + l^2 a^2} \right) \varphi_{nl} \quad (15)$$

Substituting the value of (15) in the equations (1) to (3) one obtains

$$u_x = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \frac{\lambda_l}{P} ((-1)^{m+1} + 1) \chi_{nl} \varphi_{nl} \sin(n\pi y) \quad (16)$$

$$u_y = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \frac{\lambda_l}{Q} ((-1)^{m+1} + 1) \chi_{nl} \varphi_{nl} \sin(m\pi x) \quad (17)$$

$$u_z = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \chi_{nl} \psi_{nl} \sin(m\pi x) \sin(n\pi y) \quad (18)$$

where

$$\psi_{nl} = \int_0^t \left\{ \frac{\bar{u}(m, n, t') [\cos \lambda_l \xi - \cos \lambda_l \xi (c - \xi)] - \bar{f}(m, n, t') [1 - \cos \lambda_l c]}{\cos \lambda_l \xi} \right\} \times e^{-k((m^2 + n^2)(\pi/a)^2 + (l\pi/\xi)^2)(t-t')} dt'$$

$$\text{And } \chi_{nl} = \frac{1}{\xi} \left[ \frac{8(1+\nu)k\alpha m^2}{(m^2 + n^2) + (l a/\xi)^2} \right]$$

### IV. DETERMINATION OF STRESS FUNCTIONS

Substituting the value of (15) in (11) and (12) one obtains

$$\sigma_{xx} = \sigma_{yy} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \Phi_{nl} \varphi_{nl} \sin(m\pi x) \sin(n\pi y) \quad (19)$$

$$\sigma_{zz} = \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \Phi_{nl} \varphi_{nl} \sin(m\pi x) \sin(n\pi y) \quad (20)$$

where

$$\Phi_{nl} = -\frac{8\alpha\pi Ekl}{\xi^2} \left[ \frac{l^2 a^2 + m^2 \xi^2}{(m^2 + n^2)\xi^2 + a^2} \right] \quad (21)$$

### V. DETERMINATION OF DEFLECTION

According to the classic theory of plate bending, a small deflection is defined as small compared with the plate thickness. The equation expressing the relationship between the without external load and its deflection in the case with small deflection can be defined as [4]

$$D \nabla_1^4 \omega = -\frac{\nabla_1^2 M_T}{(1-\nu)} \quad (22)$$

in which

$$\nabla_1^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}, \quad D = \frac{Et^3}{12(1-\nu^2)} \quad (23)$$

and the resultant thermal momentum is expressed as

$$M_T = \alpha E \int_0^h T z dz \quad (24)$$

where  $\omega(x, y, t)$  is the deflection of the plate midsurface,  $h$  is the plate thickness,  $D$  is flexural rigidity of plate,  $E$  is the Young's modulus,  $\nu$  is the Poisson ratio,  $\alpha$  is the thermal expansion coefficient,  $\nabla_1^4 \equiv (\nabla_1^2)^2$  is biharmonic operator and  $T$  is the temperature increment relative to some reference temperature under consideration and where stresses are zero if the plate is undeformed.

Further assuming that the edge of the square plate is fixed and clamped

$$\omega = 0, \quad \frac{\partial \omega}{\partial x} = 0 \quad \text{at } x = 0, x = a \quad (25)$$

and

$$\omega = 0, \quad \frac{\partial \omega}{\partial y} = 0 \quad \text{at } y = 0, y = a \quad (26)$$

In order to solve equation (22), we assume double trigonometric as

$$\omega(x, y, t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \varpi_{mn}(t) \sin(m\pi x) \sin(n\pi y) \quad (27)$$

Substituting equation (13) in resultant thermal momentum of equation (24), one obtains

$$M_T = \alpha E \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \left( \frac{8\pi k l}{\xi^2} \right) \varphi'_{nl} \sin(m\pi x) \sin(n\pi y) \quad (28)$$

where

$$\varphi'_{nl} = \frac{\xi}{\pi^2 \cos[l\pi]} \int_0^t \left\{ \begin{aligned} &\bar{u}(m, n, t') [hl\pi \cos[l\pi(-h+\xi)/\xi] \\ &- \sin[l\pi] + \sin[l\pi(-h+\xi)/\xi]] \\ &+ \bar{f}(m, n, t') [-hl\pi \cos[l\pi h/\xi] + \xi \sin[l\pi h/\xi]] \end{aligned} \right\} \\ \times e^{-k((m^2+n^2)(\pi/a)^2 + (l\pi/\xi)^2)(t-t')} dt'$$

From equation (24), one obtains the  $\varpi_{mn}(t)$  by substituting the equations (27) and (28) as

$$\varpi_{mn}(t) = \frac{\alpha E}{D \pi^2 (1-\nu)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \left( \frac{8\pi k l}{\xi^2 (m^2 + n^2)} \right) \varphi'_{nl} \quad (29)$$

Finally in order to obtain the required thermal deflection  $\omega(x, y, t)$ , we substitute the  $\varpi_{mn}(t)$  from equation (29) into equation (27) as

$$\omega(x, y, t) = \frac{\alpha E}{D \pi^2 (1-\nu)} \sum_{m=1}^{\infty} \sum_{n=1}^{\infty} \sum_{l=0}^{\infty} \left( \frac{8\pi k l}{\xi^2 (m^2 + n^2)} \right) \varphi'_{nl} \sin(m\pi x) \sin(n\pi y) \quad (30)$$

The results show that the small deflection theory is suitable for thermo elastic thin plates with small deflection in comparison with the plate thickness.

### VI. SPECIAL CASE

Set

$$u(x, y, t) = (1 - e^{-t})(a-x)(1 - e^{-x})(a-y)(1 - e^{-y}), \quad (31)$$

$$f(x, y, t) = (1 - e^{-t})(a-x)(1 - e^{-x})(a-y)(1 - e^{-y})(1 + e^{-\xi}) \quad (32)$$

Applying the procedure described and followed in section (13) to equations (31) and (32), one obtains

$$\bar{u}(m, n, t) = \Omega_m \Psi_n (1 - e^{-t}), \quad (33)$$

$$\bar{f}(m, n, t) = \Omega_m \Psi_n (1 - e^{-t})(1 + e^{-\xi}) \quad (34)$$

Where

$$\Omega_m = \frac{a^2}{m\pi} - \frac{m\pi}{1+(m\pi/a)^2} \{1 - (-1)^m e^{-a}\},$$

$$\Psi_n = \frac{a^2}{n\pi} - \frac{n\pi}{1+(n\pi/a)^2} \{1 - (-1)^n e^{-a}\}$$

Substituting the value of (33) and (34) in the equations (13) to (30) one obtains temperature distribution, unknown temperature gradient, Airy's stress functions, displacement functions, stress functions and deflections.

### VII. CONVERGENCE OF THE SERIES SOLUTION

In order for the solution to be meaningful the series expressed in equations (13) should converge for all  $0 \leq x \leq a$ ,  $0 \leq y \leq a$  and  $0 \leq z \leq h$ , and we should further investigate the conditions which has to be imposed on the functions  $u(x, y, t)$  and  $f(x, y, t)$  so that the convergence of the series expansion for  $T(x, y, z, t)$  is valid. The temperature equations (13) can be expressed as

$$T(x, y, z, t) = \sum_{m=1}^M \sum_{n=1}^N \sum_{l=0}^L \left( \frac{8\pi k l}{\xi^2} \right) \varphi_{nl} \times \sin(m\pi x) \sin(n\pi y) \quad (35)$$

We impose conditions so that both  $T(x, y, z, t)$  converge in some generalized sense to  $\hat{g}(x, y, z)$  respectively as  $t \rightarrow 0$  in the transform domain. Taking into account of the asymptotic behaviors of Fourier sine transform as defined in [5], it is observed that the series expansion for  $T(x, y, z, t)$  will be convergent by one term approximation as

$$\varphi_{nl} = \int_0^t \left\{ \frac{\bar{u}(m, n, t') \sin[l\pi(z-\xi)/\xi] - \bar{f}(m, n, t') \sin[l\pi z/\xi]}{\cos[l\pi]} \right\} \times e^{-k((m^2+n^2)(\pi/a)^2 + (l\pi/\xi)^2)(t-t')} dt'$$

$$= O\{1/[(\pi/a)^2((la/\xi)^2 + m^2 + n^2)]^\kappa\}, \quad \kappa > 0 \quad (36)$$

Thus,  $T(x, y, z, t)$  is convergent to a limit  $\{T(x, y, z, t)\}_{(x,y)=a, z=h}$  as convergence of a series for  $(x, y) = a$  implies convergence for all  $(x, y) \leq a$  at any value of  $z$ . But for an exact solution would require the use of an infinite number of terms in the equations. In the present solution only the first 10 terms in the series are considered. The effects of truncating of numbers are brought out by the comparison table for solutions of different functions for 5, 10, 15 and 50 terms. It is evident from the table that the convergence is rapid for the

temperature distribution and somewhat slower in the case of stresses, while it is estimated from table that the possible error from the table is less than 2 percent.

Functions	5 terms	10 terms	15 terms	50 terms
$T(x, y, z, t)$	4.12E-03	3.62E-03	3.47E-03	3.40E-03
$g(x, y, t)$	-5.13E-03	-4.51E-03	-4.33E-03	-4.24E-03
$\sigma_{xx}, \sigma_{yy}$	-3.06E+00	-2.70E+00	2.59E+00	-2.55E+00
$\sigma_{zz}$	-1.59E-01	-1.26E-01	-1.09E-01	-9.40E-03

Table 1. Convergence of solution as the number of terms used in the equation are increased from 5 to 50

### VIII. NUMERICAL RESULTS, DISCUSSION AND REMARKS

To interpret the numerical computation we consider material properties of low carbon steel (AISI 1119), which can be used for medium duty shafts, studs, pins, distributor cams, cam shafts, and universal joints having mechanical and thermal properties

$$\kappa = 13.97 [\mu m / s^2], \nu = 0.29,$$

$$\lambda = 51.9 [W / (m - K)] \text{ and}$$

$$a_t = 14.7 \mu m / m - ^0 C.$$

Setting the physical parameter with  $a = 2$ ,  $b = 2$ ,  $h = 0.50$  and  $\xi = 0.35$ . In the foregoing analysis will be illustrated by the numerical results shown in Figure 2 to 12. Figure 2 depicts the distributions of the temperature increment  $T(x, y, z, t)$  verse thickness at different values of time with interior point  $\xi = 0.35$ . It shows that heat gain on boundary  $z = 0$  is zero and then initially temperature increment increases slowly with increase of distance along  $z$ -axis and the physical meaning emphasis for this phenomenon is that there is increment in the rate of heat propagation initially and reaches maximum then drops unto the known interior point under consideration. It is also observed that this dropping nature crosses the inner core at the interior point  $\xi = 0.35$  and approaches towards outer edge leading to compressive stress on the outer part. Figure 3 depicts the displacement function. Figure 4 and 5 shows the distributions of the thermal stresses at different value of time. The stresses  $\sigma_{zz}$  is smaller than stress  $\sigma_{xx} = \sigma_{yy}$  can be seen from figures 4-5. Figure 6-8 depicts the thermal deflection showing the sinusoidal nature with increasing trend of peak with the fixed value of time. Figure 9-12 depicts the temperature increment  $T(x, y, z, t)$  verse time at different interior point.

**IX. COMPARISON WITH THE RESULTS OBTAINED BY PREVIOUS AUTHOR**

The earlier work on the thermo elastic problem of the clamped rectangular plate known to the author is that in 2003 by Noda et. al. [4]. Previous author solved the direct transient thermo elastic problem using trigonometric series and made a thorough analysis of the moments and deflections for the rectangular plate with deflection. The results for the inverse thermo elastic problem of the square plate with clamped edge closely agrees with results presented in [4] and only difference lies in the peak. As would be expected, the square plate is more rigid than the long rectangular plate.

**X. CONCLUSION**

In this problem, we modify the conceptual ideal proposed by Durge et. al (2005) for thermo elastic problem for rectangular plate and Adams et. al (1999) for laminated rectangular plate. The temperature distributions, displacement and stress functions at the edge  $z = h$  occupying the region of the flat square plat with  $a$  as the length of sides and  $h$  as the small thickness have been obtained with the known source function. We develop the analysis for the temperature field for heating process by introducing the transformation finite Fourier sine transform and Laplace transform techniques with prescribed boundary conditions. The series solutions converge provided we take sufficient number of terms in the series. Since the thickness of square is very small, the series solution given here will be definitely convergent. Any particular case can be derived by assigning suitable values to the parameters and functions in the series expressions. The temperature, displacement and thermal stresses that are obtained can be applied to the design of useful structures or machines in engineering applications.

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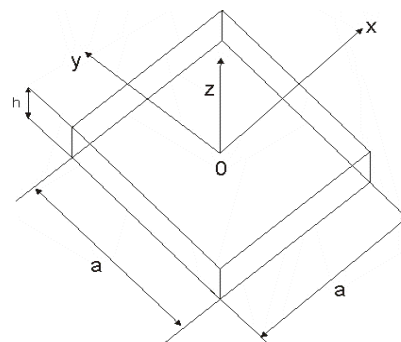


Fig (1).The configuration of thin square plate

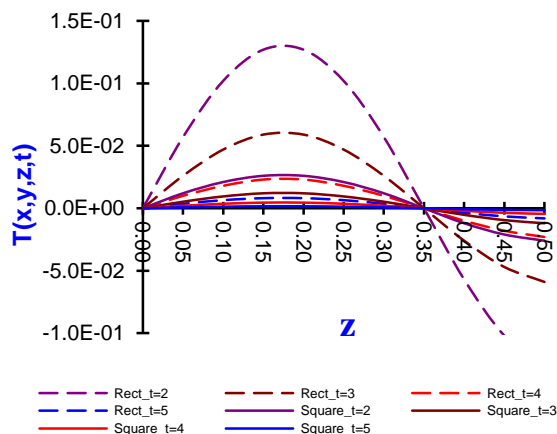


Fig (2).Distribution of the temperature versus thickness for rectangular and square plate



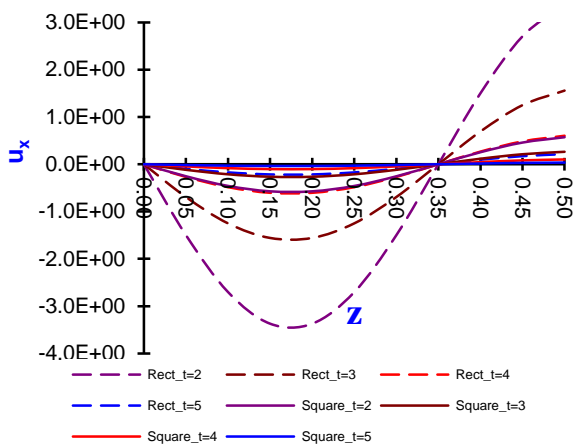


Fig (3). Distribution of the displacement function versus thickness for rectangular and square plate

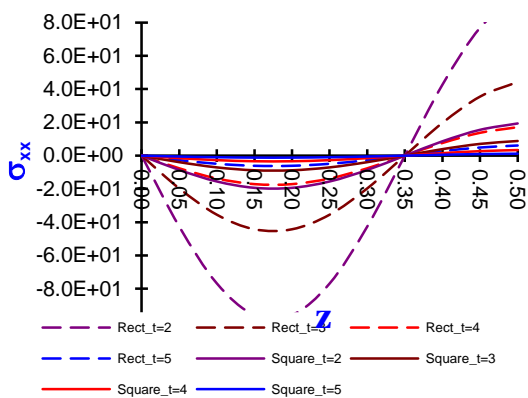


Fig (4). Distribution of the stress function versus thickness for rectangular and square plate

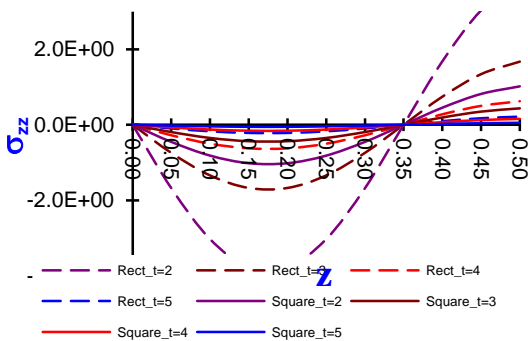


Fig (5). Distribution of the stress function versus thickness for rectangular and square plate

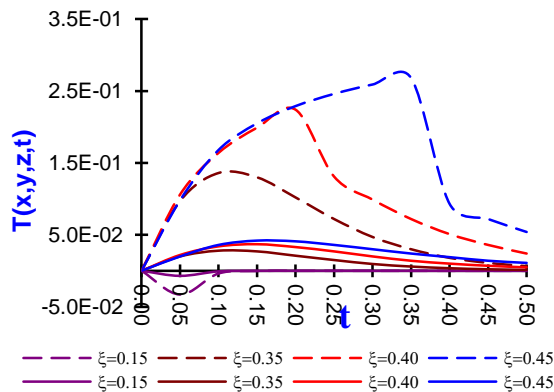


Fig (6). Distribution of the temperature versus time for rectangular and square plate

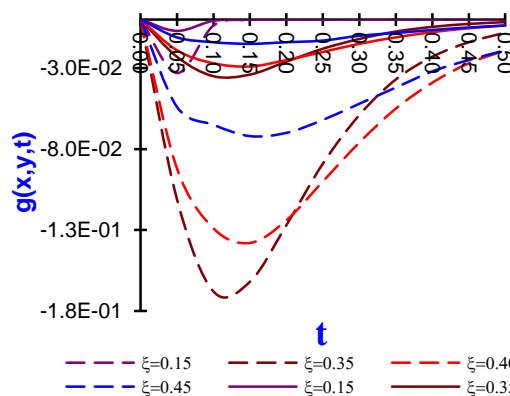


Fig (7). Distribution of the unknown temperature gradient versus time for rectangular and square plate

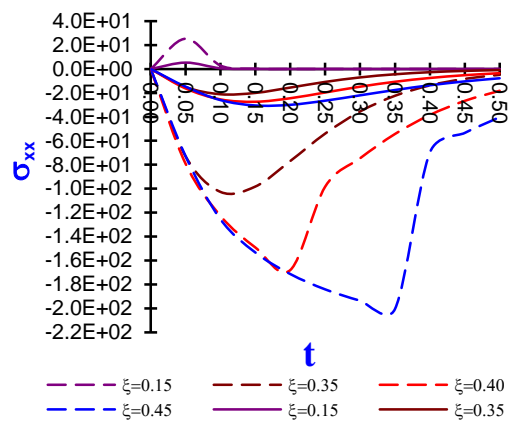


Fig (8). Distribution of the stress function versus time for rectangular and square plate

**AUTHOR BIOGRAPHY**



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