ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 3, Issue 4, October 2013

Thermoelastic Analysis of a Thin Circular Plate with Radiation Type Conditions

Hamna Parveen; Anjali C. Pathak; Varghese, V. and Khobragade, N. W.
Post Graduate Department of Mathematics, MJP Educational Campus,
RTM Nagpur University, Nagpur – 440 033, India

Abstract- This paper deals with the study of temperature distribution, stresses and deflections of a thin circular plate clamped with ring producing small deflection subjected to partially heat distribution heat supply. Values of the radial and axial stress functions have been obtained on the surface of a circular plate. The results have been compared with the previous analysis known to the author and found in agreement with it.

Key words: Transient response, round plate, temperature distribution, thermal stress, deflection, integral transform

I. INTRODUCTION

Roy Choudhari [6] successfully investigated the quasi-static thermal stresses in thin circular plate due to transient temperature applied along the circumference of a circle over the upper face. Nowacki [2] determined the quasi-static thermal stresses in a thick circular plate due to a temperature field. Noda et. al. [1] has considered a circular plate and discussed the transient thermo elastic-plastic bending problem, making use of the strain increment theorem. Deshmukh et. al. [5] has succeeded in determining the thermal deflection at the center over a circular plate applying finite Hankel and Fourier transform using dirichlet type of boundary conditions. The earliest work on the problem of the clamped circular plate over a thick disc known to author is that by Khobragade et. al. [4].

Here an attempt is made to determine the temperature distribution, radial and axial stress functions of a round plate, while deflection function is analyzed at the center of the clamped round plate by ring with the stated boundary conditions subjected to known partially heat supply by using Integral transform techniques.

II. FORMULATION OF THE PROBLEM: GOVERNING EQUATION

Consider a round plate of small thickness h occupying the space $D=\{(x,y,z)\in R^3 : 0\le (x^2+y^2)^{1/2}\le a,\ 0\le z\le h\}$ and for small thickness in a plane state of stress, the differential equation governing the displacement function $\phi(r,z,t)$, where $r=(x^2+y^2)^{1/2}$, for the heating as [3]

$$\frac{\partial^2\phi}{\partial r^2}+\frac{1}{r}\frac{\partial\phi}{\partial r}=(1+\nu)a_t\,T \tag{1}$$

with

$$\phi=0 \text{ at } r=a \text{ for all time } t \tag{2}$$

where ν and a_t are the Poisson's ratio and the linear coefficient of thermal expansion of the material of the disc respectively and $T(r,z,t)$ is the heating temperature of the disc at time t satisfying the differential equation

$$\kappa\left[\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial T}{\partial r}\right)+\frac{\partial^2 T}{\partial z^2}\right]=\frac{\partial T}{\partial t} \tag{3}$$

Where $\kappa=K/\rho c$ is the thermal diffusivity of the material of the disc, K is the conductivity of the medium, c is its specific heat and ρ is its calorific capacity (which is assumed to be constant), subject to the initial and boundary conditions

$$M_t(T,\bar{k}_1,0,0)=0 \text{ for all } 0\le r\le a,\ 0\le z\le h \tag{4}$$

$$M_r(T,\bar{k}_1,0,a)=(-Q_0/\lambda)f(z,t) \text{ for all } 0\le z\le h,\ t>0 \tag{5}$$

$$M_z(T,0,1,0)=F_1(r,t), \text{ for all } 0\le r\le a,\ t>0 \tag{6}$$

$$M_z(T,0,\bar{k}_2,h)=F_2(r,t) \quad \text{for all } 0\le r\le a,\ t>0 \tag{7}$$

being:

$$M_\vartheta(f,\bar{k}_1,\bar{k}_2,\$)=(\bar{k}_1 f+\bar{k}_2\hat{f})_{\vartheta=\$}$$

where the prime (^) denotes differentiation with respect to ϑ, Q_0 is the heat flux, radiation constants are $\bar{k}_1$ and $\bar{k}_2$ on the curved surfaces of the plate respectively. For convenience we consider $\bar{k}_1=\bar{k}_2=1$. The functions $F_1(r,t)$ and $F_2(r,t)$ are known constants and they are set to be zero here as in other literatures [2, 6, 8] so as to obtain considerable mathematical simplicities.

The stress distribution components σ_{rr} and $\sigma_{\theta\theta}$ of the plate are given by [5]

$$\sigma_{rr}=-2\mu\frac{1}{r}\frac{\partial\phi}{\partial r} \tag{8}$$

$$\sigma_{\theta\theta}=-2\mu\frac{\partial^2\phi}{\partial r^2} \tag{9}$$

where μ is the Lame's constants, while each of the stress functions σ_{rz}, σ_{zz} and $\sigma_{\theta z}$ are zero within the disc in the plane state of stress.

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 3, Issue 4, October 2013

Further a ring of negligible thickness is clamped on the curved surface of the plate as shown in Figure 1. The differential equation satisfied by the transverse deflection of the plate center surface $\omega(r,t)$ for heating processes subjected to partial heat supply is given by [3]

$$D\nabla^4 \omega(r,t) = -\frac{\nabla^2 M_T(r,t)}{(1-\nu)} \tag{10}$$

with

$$\omega(r,t)\big|_{r=a} = 0,\ \omega'(r,t)\big|_{r=a} = 0 \tag{11}$$

where the prime (´) denotes differentiation with respect to r, $M_T(r,t)$ is the thermal moment of the plate, ν is Poisson's ratio of the plate material, D is flexural rigidity of the plate denoted by

$$D = \frac{E h^3}{12(1-\nu^2)} \tag{12}$$

E is the Young modulus, and

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} \tag{13}$$

with thermal moment defined as

$$M_T(r,t) = \alpha E \int_0^h z T(r,z,t)dz \tag{14}$$

Thus, the equations (1) to (14) constitute the mathematical formulation of the problem under consideration.

III. SOLUTION OF THE HEAT CONDUCTION EQUATION

Determination of the temperature distribution $T(r,z,t)$:

Applying finite Fourier cosine transform [7] to the equations (3), (4) and (5) over the variable r with responds to the boundary conditions (6) and taking the Laplace transform [7], one obtains $\bar{T}_C^{\,*}(r,m,s) = (-Q_0/\lambda)\bar{f}_C^{\,*}(m,s)\ (I_0(qr)/I_0(qh))$, where $q^2 = m^2\pi^2/h^2 + (s/\kappa)$ with constants complied with boundary condition (5) taking assumption that as $r \to 0$, $K_0(qr) \to \infty$, but the physical consideration of the problem remains finite; therefore one of the constant is considered as zero. Applying the inversion theorems of transform [7] and inverse Laplace transform by means of complex contour integration and the residue theorem, one obtains the expressions of the temperature distribution $T(r,z,t)$ as

$$T(r,z,t) = \left(\frac{-2Q_0}{\lambda h}\right)\chi_{mn} \times \sum_{m=1}^{\infty}\frac{\cos(m\pi z)}{h}\sum_{n=1}^{\infty}\left(\frac{2k}{a}\right)\lambda_n\left(\frac{J_0(\mu_n r)}{J_1(\mu_n a)}\right) \tag{15}$$

where

$$\chi_{mn} = \int_0^t \bar{f}_C(m,t')e^{-k(m^2\pi^2/h^2+\lambda_n^2)(t-t')}\,dt' \tag{16}$$

where $\bar{f}_C$ denotes the Fourier cosine transform of f, m is the Fourier cosine transform parameter, s is a Laplace transform parameter and μ_n is the n^{th} positive root of the transcendental equation $J_0(\mu a) = 0$.

Determination of displacement and stress function

Substituting the value of $T(r,z,t)$ from equation (15) in equation (1) and using the well known standard result

$$\left(\frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r}\right)J_0(\xi_n r) = -\xi_n^2 J_0(\xi_n r) \tag{17}$$

one obtains the thermo elastic displacement function as

$$\phi(r,z,t) = (1+\nu)a_t\left(\frac{2Q_0}{\lambda h}\right)\chi_{mn} \times \sum_{m=1}^{\infty}\frac{\cos(m\pi z)}{h}\sum_{n=1}^{\infty}\left(\frac{2k}{a}\right)\frac{1}{\lambda_n}\left(\frac{J_0(\lambda_n r)}{J_1(\lambda_n a)}\right) \tag{18}$$

Substituting the value of equation (18) in equations (8) and (9) and using the well known standard results

$$\frac{\partial}{\partial r}(J_0(\xi_n r)) = -\xi_n J_1(\xi_n r) \tag{19}$$

and

$$\frac{\partial^2}{\partial r^2}(J_0(\xi_n r)) = -\xi_n^2\left(J_0(\xi_n r) - \frac{J_1(\xi_n r)}{\xi_n r}\right) \tag{20}$$

one obtains the expression of the stress functions as

$$\sigma_{rr} = -\frac{4\mu(1+\nu)Q_0 a_t}{\lambda h}\sum_{m=1}^{\infty}\frac{\cos(m\pi z)}{h}\sum_{n=1}^{\infty}\left(\frac{2k}{a}\right)\left(\frac{J_0(\lambda_n r)}{J_1(\lambda_n a)}\right)\chi_{mn} \tag{21}$$

$$\sigma_{\theta\theta} = \frac{4\mu(1+\nu)Q_0 a_t}{\lambda h}\chi_{mn} \times \sum_{m=1}^{\infty}\frac{\cos(m\pi z)}{h}\sum_{n=1}^{\infty}\left(\frac{2k}{a}\right)\left(\frac{\lambda_n[J_0(\lambda_n r) - J_2(\lambda_n r)]}{J_1(\lambda_n a)}\right) \tag{22}$$

Determination of transverse deflection

In solving equations (10) for a round plate clamped with ring of negligible thickness on the curved surface, we assume the unknown deflection $\omega(r,t)$ for the center of the plate satisfying equation (10) as

$$\omega(r,t) = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty} C_{mn}(t)[J_0(\lambda_n r) - J_0(\lambda_n a)] \tag{23}$$

Substituting equation (15) in resultant thermal momentum defined in (14), one obtains

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 3, Issue 4, October 2013

$$M_T(r,t)=\alpha E\left(\frac{-2Q_0}{\lambda h}\right)\chi_{mn}$$
$$\times\sum_{m=1}^{\infty}\left(\frac{-1+\cos(m\pi h)+m\pi h\sin(m\pi h)}{m^2h^2}\right)\sum_{n=1}^{\infty}\left(\frac{2k}{a}\right)\lambda_n\left(\frac{J_0(\mu_n r)}{J_1(\mu_n a)}\right) \tag{24}$$

Substituting equation (23) and (24) in equation (10), $C_{mn}(t)$ is obtained as

$$C_{mn}(t)=\alpha E\left(\frac{4\alpha E Q_0 k}{a\lambda h^3}\right)\chi_{mn}$$
$$\times\sum_{m=1}^{\infty}\left(\frac{-1+\cos(m\pi h)+m\pi h\sin(m\pi h)}{\lambda_n m^2 J_1(\mu_n a)}\right) \tag{25}$$

Finally in order to obtain the required thermal deflections $\omega(r,t)$, we substitute the $C_{mn}(t)$ in equation (10) as

$$\omega(r,t)=\frac{4\alpha E Q_0 k}{a\lambda h^3}$$
$$\times\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{-1+\cos(m\pi h)+m\pi h\sin(m\pi h)}{\lambda_n m^2 J_1(\mu_n a)}\right)$$
$$\times[J_0(\lambda_n r)-J_0(\lambda_n a)]\chi_{mn} \tag{26}$$

IV. SPECIAL CASE

Set

$$f(z,t)=(1-e^{-t})(h-z^2/2)e^{-h} \tag{27}$$

Applying finite Fourier cosine to the equation (27) one obtains

$$\bar{f}_C(m,t)=\int_0^h(1-e^{-t})(h-z^2/2)e^{-h}h^{-1}\cos(m\pi z)\,dz \tag{28}$$

Using equation (28) in the equations (15), (18), (21), (22) and (23), one obtains

$$\frac{T(r,z,t)}{A}=\sum_{m=1}^{\infty}\cos(m\pi z)\sum_{n=1}^{\infty}\lambda_n\left(\frac{J_0(\mu_n r)}{J_1(\mu_n a)}\right)\hat{\chi}_{mn} \tag{29}$$

$$\frac{\phi(r,z,t)}{B}=\sum_{m=1}^{\infty}\cos(m\pi z)\sum_{n=1}^{\infty}\frac{1}{\lambda_n}\left(\frac{J_0(\lambda_n r)}{J_1(\lambda_n a)}\right)\hat{\chi}_{mn} \tag{30}$$

$$\frac{\sigma_{rr}}{C}=-\sum_{m=1}^{\infty}\cos(m\pi z)\sum_{n=1}^{\infty}\left(\frac{J_0(\lambda_n r)}{J_1(\lambda_n a)}\right)\hat{\chi}_{mn} \tag{31}$$

$$\frac{\sigma_{\theta\theta}}{C}=\sum_{m=1}^{\infty}\cos(m\pi z)\sum_{n=1}^{\infty}\left(\frac{\lambda_n[J_0(\lambda_n r)-J_2(\lambda_n r)]}{J_1(\lambda_n a)}\right)\hat{\chi}_{mn} \tag{32}$$

$$\frac{\omega(r,t)}{D}=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{-1+\cos(m\pi h)+m\pi h\sin(m\pi h)}{\lambda_n m^2 J_1(\mu_n a)}\right)$$
$$\times[J_0(\lambda_n r)-J_0(\lambda_n a)]\hat{\chi}_{mn} \tag{33}$$

Where

$$\hat{\chi}_{mn}=e^{-h-(1+\kappa m^2\pi^2/h^2+\kappa\lambda_n^2)t}(e^t-1)$$
$$\times(1-e^{k(m^2\pi^2/h^2+\lambda_n^2)t})$$
$$\times\left(\frac{2hm\pi\cos(hm\pi)+(h^2m^2\pi^2-2-2hm^2\pi^2)\sin(hm\pi)}{2hm^3\pi^3\kappa(m^2\pi^2/h^2+\lambda^2)}\right),$$

$A=\dfrac{-4kQ_0}{a\lambda h^2}$, $B=\dfrac{4k(1+v)a_tQ_0}{a\lambda h^2}$, $C=\dfrac{8\mu\kappa(1+v)Q_0a_t}{a\lambda h^2}$, $D=\dfrac{4\alpha E Q_0 k}{a\lambda h^3}$.

V. CONVERGENCE OF THE SERIES SOLUTION

In order for the solution to be meaningful the series expressed in equations (15) should converge for all $a\le r\le b$ and $0\le z\le h$, and we should further investigate the conditions which has to be imposed on the functions $f(z,t)$ so that the convergence of the series expansion for $T(r,z,t)$ is valid. The temperature equations (15) can be expressed as

$$T(r,z,t)=\left(\frac{-2Q_0}{\lambda h}\right)\sum_{m=1}^{M'}\frac{\cos(m\pi z)}{h}$$
$$\times\sum_{n=1}^{\infty}\left(\frac{2k}{a}\right)\lambda_n\left(\frac{J_0(\mu_n r)}{J_1(\mu_n a)}\right)\chi_{mn} \tag{34}$$

We impose conditions so that both $T(r,z,t)$ converge in some generalized sense to $g(r,s)$ as $t\to 0$ in the transform domain. Taking into account of the asymptotic behaviors of transform as given in [7], it is observed that the series expansion for both $T(r,z,t)$ will be convergent by one term approximation as

$$\chi_{mn}=\int_0^t\bar{f}_C(m,t')e^{-k(m^2\pi^2/h^2+\lambda_n^2)(t-t')}dt', \quad \kappa>0$$
$$=O\left\{1/\left(\mu_n^2+m^2\pi^2/h^2\right)^{\kappa}\right\} \tag{35}$$

Here $\bar{f}_C(m,t')$ in equation (28) can be chosen as one of the following functions or their combination involving addition or multiplication of constant, $\sin(\omega t)$, $\cos(\omega t)$,

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 3, Issue 4, October 2013

$\exp(kt)$, or *polynomials in* t. Thus, $T(r, z, t)$ is convergent to a limit $\{T(r, z, t)\}_{r=b, r=h}$ as convergence of a series for $r = b$ implies convergence for all $r \le b$ at any value of z.

VI. NUMERICAL RESULTS, DISCUSSION AND REMARKS

To interpret the numerical computation we consider material properties of low carbon steel, which can be used for medium duty shafts, studs, pins, distributor cams, cam shafts, and universal joints having mechanical and thermal properties $\kappa = 13.97$ $\upsilon = 0.29$, $\lambda = 51.9$ and $a_t = 14.7$. With the general convention that the thicknesses of the thin round plate is taken $\le$ (diameter / 40) as $h = 0.1$, with radius $a = 1$. In the foregoing analysis will be illustrated by the numerical results shown in Figure 2 to 12. Figure 2 depicts the distributions of the temperature increment $T(r, z, t)$ verse radius and thickness at a fixed value of time $t = 2$. It shows that heat gain follows increasing trend of sinusoidal nature with increase of radius up to the outer region of radiation flux. The physical meaning emphasis for this phenomenon is that there is increment in the rate of heat propagation with radius which leads to compressive radial stress at inner part and expand more on outer due to partially distributed annular heat supply. Figure 2 depicts the displacement function and it is noteworthy that it is in agreement with the boundary condition (2) and attains zero at the outer edge. Figure 3 and 4 shows the distributions of the radial and axial thermal stresses at fixed value of time. Figure 10 shows the deflection trend with same parameter aforementioned. All other figures are showing similar character & self-explanatory.

VII. CONCLUSION

In this problem, we modify the conceptual ideal proposed by Noda [6] for circular plate and investigated further for the temperature distributions, displacement, stress function and deflection. As a special case mathematical model is constructed and performed numerically. We develop the analysis for the temperature field for heating processes by introducing the temperature function satisfying all boundaries conditions of radiations type. The series solutions converge provided we take sufficient number of terms in the series. Since the thickness of round plate is very small, the series solution given here will be definitely convergent. Assigning suitable values to the parameters and functions in the series expressions can derive any particular case of usage. The temperature, displacement and thermal stresses that are obtained can be applied to the design of useful structures or machines in engineering applications.

ACKNOWLEDGMENT

The authors are thankful to University Grant Commission, New Delhi to provide the partial financial assistance under **major research project** scheme.

REFERENCES

[1] Ishihara, M., Tanigawa, Y., Kawamura, R., and Noda, N., Theoretical Analysis of thermoelastic-plastic deformation of a circular plate due to a partially distributed heat supply, Journal of Thermal Stresses Vol. 20, No. 2, pp. 203-233 (1997).

[2] Khobragade, K. W., Varghese, V. and Khobragade, N. W., An inverse transient thermoelastic problem of a thin annular disc, Applied Mathematics E-Notes, Vol. 3, pp 617-625 (2006).

[3] Khobragade, N. L., and Deshmukh, K. C., Thermoelastic problem of a thin circular plate subjected to a distributed heat supply, Journal of Thermal Stresses Vol. 28, pp. 171-184 (2005).

[4] Durge, M. H. and Khobragade, N. W., Deflection of a clamped circular plate on a solid circular cylinder, Bull. Of the Calcutta Math. Soc. (2002).

[5] Nowacki, W., The state of stress in a thick Circular plate due to temperature field, Bull. Sci. Acad. Polon Sci. Tech., Vol V, pp 227 (1957)

[6] Roy Choudhari, S. K., A note on the quasi-static stress in a thin circular plate due to transient temperature applied along the circumference of a circle over the upper face, bull. Acad. Polon. Sci. Ser., Sci. Tech No. 1, pp 20-24 (1972)

[7] Sneddon, I. N., Fourier Transform, McGraw Hill Book Company (1951)

[8] Wankhede, P.C., On the quasi-Static thermal stresses in a circular plate; Indian J. pure & Applied Math, Vol. 13 (11), pp.1273-1277 (1982)

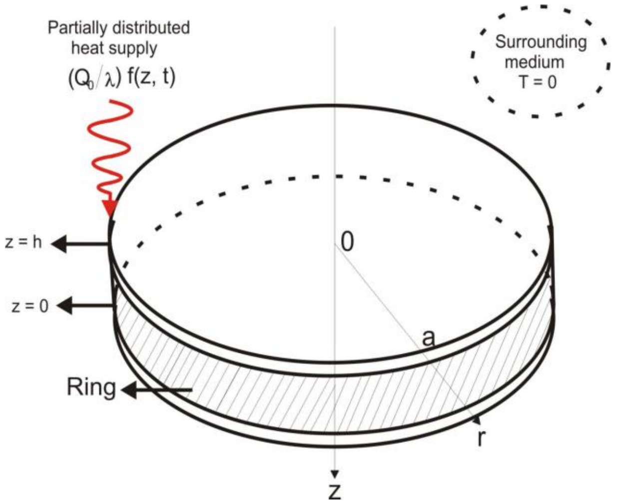


Fig (1).The configuration of thin round plate clamped with ring

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 3, Issue 4, October 2013

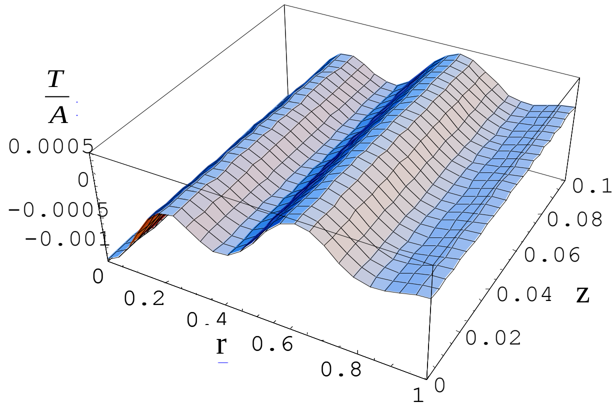


Fig (2) .Distribution of the temp. versus *r* and *z*-axis for *t*=0.2

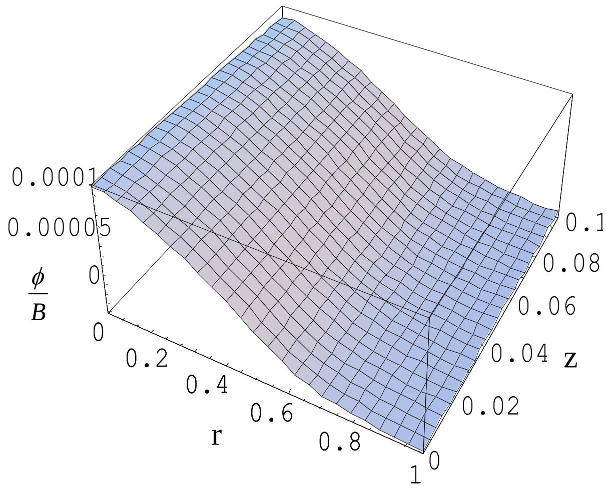


Fig (3).Distribution of the displacement function versus *r* and *z*-axis for *t*=0.2

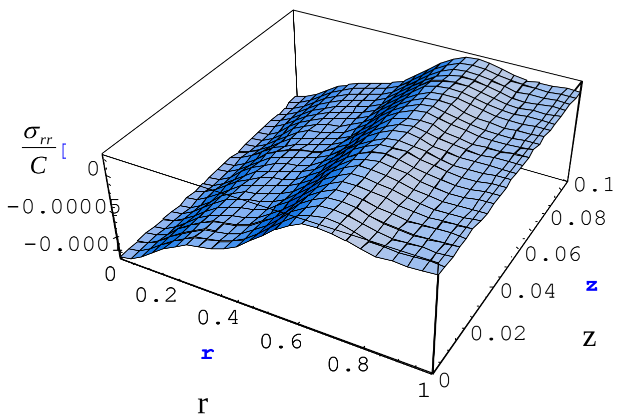


Fig (4).Distribution of the radial stress function versus *r* and *z*-axis for *t*=0.2

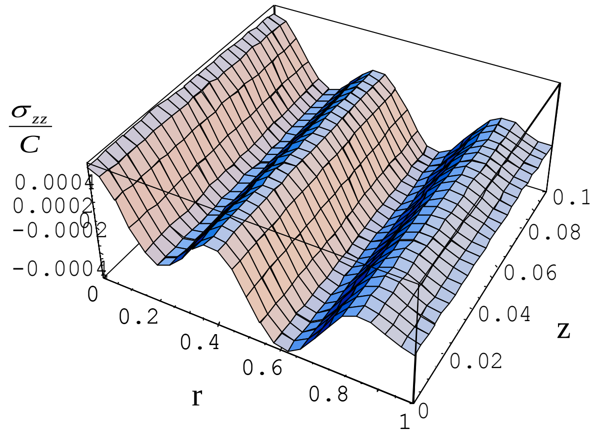


Fig (5).Distribution of the axial stress function versus *r* and *z*-axis for *t*=0.2

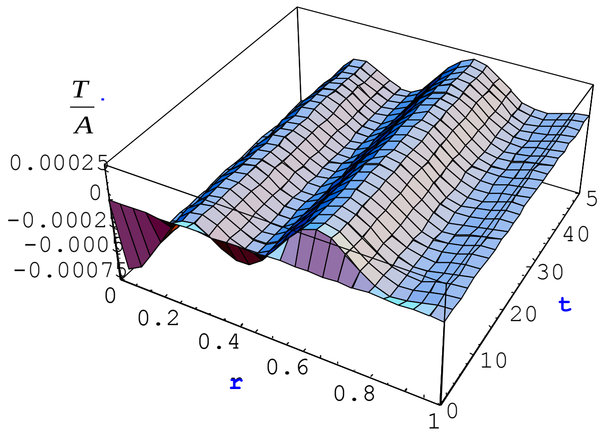


Fig (6).Distribution of the temperature versus *r* and *t* for *z*=0.05

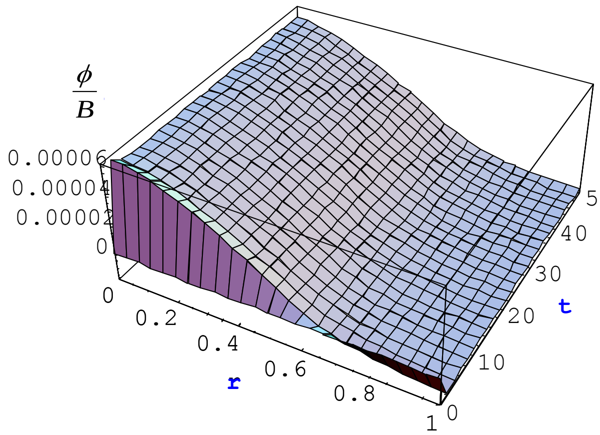
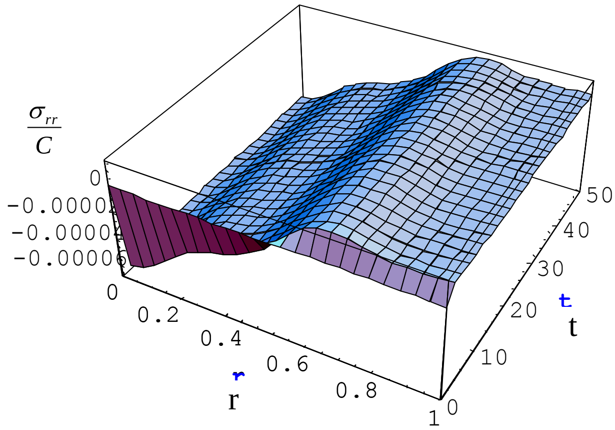


Fig (7).Distribution of the displacement function versus *r* and *t* for *z*=0. 05

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 3, Issue 4, October 2013



Fig(8).Distribution of the radial stress function versus *r* and *t* for *z*=0. 05

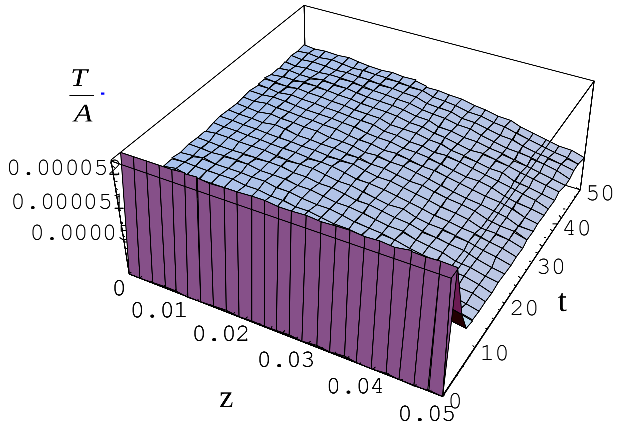


Fig (11), Distribution of the temperature versus *t* and *z*-axis for *r*=0.75

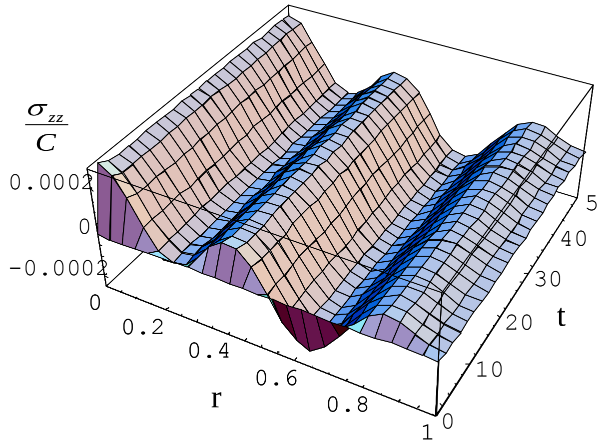


Fig (9).Distribution of the axial stress function versus *r* and *t* for *z*=0.05

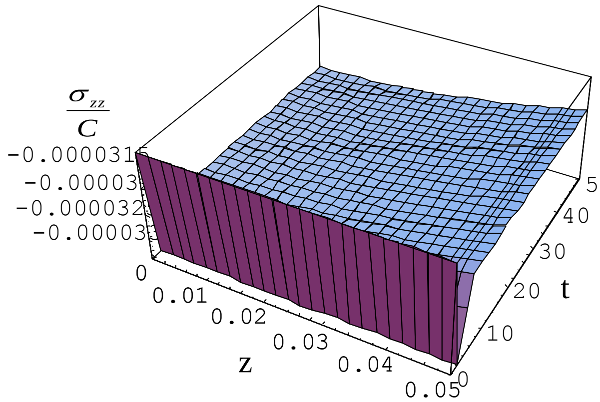


Fig (12).Distribution of the axial stress Function versus *t* and *z*-axis for *r*=0.75

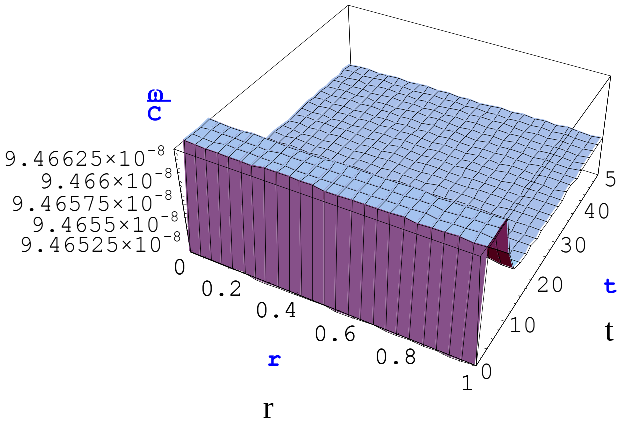


Fig (10).Deflection function versus *r* and *t* for *z*=0.05

AUTHOR BIOGRAPHY

Dr. N.W. Khobragade For being M.Sc in statistics and Maths he attained Ph.D. He has been teaching since 1986 for 27 years at PGTD of Maths, RTM Nagpur University, Nagpur and successfully handled different capacities. At present he is working as Professor. Achieved excellent experiences in Research for 15 years in the area of Boundary value problems and its application. Published more than180 research papers in reputed journals. Fourteen students awarded Ph.D Degree and four students submitted their thesis in University for award of Ph.D Degree under their guidance.