Abstract— Adjacent vertex distinguishing edge colouring or $\alpha\text{vdl}$ -colouring of a graph $G$ is a mapping from its edge set to the set of nonnegative integers such that no pair of adjacent vertices meet the same set of colours. The $\alpha\text{vdl}$ -chromatic number, denoted by $\chi'_a(G)$, is the minimum number of colours needed in an $\alpha\text{vdl}$ -colouring of $G$. A cactus graph is a connected graph in which every block is either an edge or a cycle and in other words, no edge belongs to more than one cycle. Here is proved that for a cactus graph $G$, $\Delta \leq \chi'_a(G) \leq \Delta + 3$, where $\Delta$ is the degree of $G$. An optimal algorithm is also presented to colour the edges using $\alpha\text{vdl}$ -edge colouring technique on cactus graphs in $O(n)$ time, where $n$ is the total number of vertices of the cactus graph.

Keywords:-- adjacent vertex distinguishing edge chromatic number, adjacent vertex distinguishing edge colouring, analysis of algorithms, cactus graph, design of algorithms, graph colouring

I. INTRODUCTION

Cactus graph is a connected graph in which every block is a cycle or an edge, in other words, no edge belongs to more than one cycle. Cactus graph have extensively studied and used as models for many real world problems. This graph is one of the most useful discrete mathematical structure for modelling problem arising in the real world. It has many applications in various fields like computer scheduling, radio communication system, etc. Cactus graph have studied from both theoretical and algorithmic points of view. This graph is a subclass of planar graph and superclass of tree. Let $G$ be a simple graph with $n$ vertices. For $d \geq 0$ write $n_d$ for the number of vertices of $G$ of degree $d$. Let $\chi'(G)$ be the minimum number of colours required in a proper edge-colouring of $G$. By Vizing’s theorem we know that $\Delta \leq \chi'(G) \leq \Delta + 1$. A proper edge colouring is said to be vertex-distinguishing if each pair of vertices is incident to a different set of colours. Suppose that $G=(V,E)$ is a graph and $\phi : E \to \{c_1, c_2, \ldots, c_k\}$ is a proper edge colouring of $G$. For any vertex $v \in V$, let $d_{\phi}(v)$ or simply $d(v)$ denote degree of $v$ in $G$ and $\Phi(v) = \{\phi(vw) \mid vw \in E\}$. If $uv \in E$ then $u$ is called a neighbor of $v$ and $v$, a neighbor of $u$. We say a colour $c_i$ is incident with a vertex $u \in V(G)$, if there exists an edge $(u,v)$ is coloured by $c_i$. A proper edge colouring $\phi$ is called an adjacent vertex distinguishing edge colouring or $\alpha\text{vdl}$ -edge colouring if $\Phi(u) \neq \Phi(v)$ for all $uv \in E$. The vertex distinguishing proper edge colouring will also be called as strong edge colouring. It is clear that every graph without isolated edges has an $\alpha\text{vdl}$ -colouring. A $k$- $\alpha\text{vdl}$ -colouring is an $\alpha\text{vdl}$ -colouring using at most $k$ colours. The $\alpha\text{vdl}$-chromatic number of $G$, denoted by $\chi'(G)$, is the minimum number of colours needed in an $\alpha\text{vdl}$-colouring of $G$. The concept of vertex-distinguishing edge colouring has been considered in several papers [1, 4, 5, 6, 7, 11, 12]. Zhang et al. [15] presented the following conjecture.

Conjecture 1 [15] If $G$ be a simple connected graph with at least three vertices and $G \neq C_3$, then $\Delta \leq \chi'_a(G) \leq \Delta + 2$.

II. REVIEW OF PREVIOUS WORK

Several results are known for adjacent vertex distinguishing edge colouring of graphs, but to the best of our knowledge no result is known for cactus graph. In this section, the known results for general graphs and some related graphs of cactus graphs are presented. A $\chi'_a(G)$ is at least as large as the edge chromatic number of $G$, it is clear that $\chi'_a(G) \geq \Delta$. Balister et al. [3] proved Conjecture 1 for all graphs with $\Delta = 3$, and for all bipartite graphs. They also showed that the bound is tight. Only much weaker bounds are known for general graphs without any isolated edges. Akbari et al. [2] obtained the bound $\chi'_a(G) \leq 3\Delta$ for all graphs without any isolated edges. For very large $\Delta$, Hatami [10] proved that $\chi'_a(G) \leq \Delta + 300$ if $\Delta > 10^{10}$, and Ghandehari and Hatami [9] proved that $\chi'_a(G) \leq \Delta + 27\sqrt{\Delta n} \Delta$ if $\Delta > 10^6$. Recently, Edwards et al. [8] proved that if $G$ is a planar bipartite graph with $\Delta(G) \geq 12$, the $\chi'_a(G) \leq \Delta + 1$. In [13], Liu and Liu proved that for any connected 3-colorable Hamiltonian graph $G$, $\chi'_a(G) \leq \Delta + 3$. In [15], Zhang et al. find out the result for complete
III. THE AVD-COLOURING OF INDUCED SUB-GRAPHS OF CACTUS GRAPHS

Let $G = (V, E)$ be a given graph and subset $U$ of $V$ the induced subgraph by $U$, denoted by $G[U]$, is the given graph $G' = (U, E')$, where $E' = \{(u, v) : u, v \in U \text{ and } (u, v) \in E\}$.

![Fig. 1: Induce subgraphs of cactus graph.](image)

The cactus graph have many interested subgraphs, those and their *avd-edge colouring* are illustrated below. An edge is denoted by $P_2$, so $\chi'(an\ edge) = 1$. The star graph $K_{1,\Delta}$ is a subgraph of $K_{n,\Delta}$ therefore one can conclude the following result.

**Lemma 1** For any star graph $K_{1,\Delta}$, $\chi'_n(K_{1,\Delta}) = \Delta$, where $\Delta$ is the degree of the star graph.

III. AVD-EDGE COLOURING OF CYCLES

A. Avd-edge colouring of one cycle

In [15], Zhang et al. have colour $C_n$ by *avd-edge colouring* and they have obtained the following result. Here we have given a constructive prove of this result.

**Lemma 2** [15] For any cycle $C_n$ of length $n$,

$$\chi'_n(C_n) = \begin{cases} 3, & \text{if } n = 0 \pmod{3}; \\ 4, & \text{if } n = 1 \pmod{3}; \\ 5, & \text{if } n = 2 \pmod{3}. \end{cases}$$

**Proof:** We assume that $C_n$ is a cycle of length $n$. Let $v_i$'s and $e_i$'s; $i = 0, 1, \ldots, n-1$ be the vertices and edges of $C_n$ respectively. To colour the edges of the cycle by *avd-colouring* we classify the cycle into three groups, viz., $C_{3k}$, $C_{3k+1}$ and $C_{3k+2}$ respectively. Here $e_0 = (v_0, v_1)$, $e_{n-1} = (v_{n-1}, v_0)$ and $e_i = (v_i, v_{i+1})$ for $i = 1, 2, \ldots, n-2$. Now we colour the edges of $C_n$ as follows.

**Case 1:** If $n = 3k = 0 \pmod{3}$.

Then we colour the edges as

$$\phi(e_i) = \begin{cases} 0, & \text{if } i = 0 \pmod{3}; \\ 1, & \text{if } i = 1 \pmod{3}; \\ 2, & \text{if } i = 2 \pmod{3}. \end{cases}$$

here $\Phi(v_i) = \begin{cases} [2,0], & \text{if } i = 0 \pmod{3}; \\ [0,1], & \text{if } i = 1 \pmod{3}; \\ [1,2], & \text{if } i = 2 \pmod{3}. \end{cases}$

**Case 2:** If $n = 3k + 1 = 1 \pmod{3}$.

Here we colour the first $3k$ edges of $C_{3k+1}$ as

$$\phi(e_i) = \begin{cases} 0, & \text{if } i = 0 \pmod{3}; \\ 1, & \text{if } i = 1 \pmod{3}; \\ 2, & \text{if } i = 2 \pmod{3}. \end{cases}$$

And remaining edges as

$$\phi(e_i) = \begin{cases} 3, & \text{if } i = 3k+1 \text{ to } 3k+3 \pmod{3}. \end{cases}$$

$$\Phi(v_i) = \begin{cases} [0,1], & \text{if } i = 1 \pmod{3}; \\ [1,2], & \text{if } i = 2 \pmod{3}; \\ [2,0], & \text{if } i = 0 \pmod{3}. \end{cases}$$

**Case 3:** If $n = 3k + 2 = 2 \pmod{3}$.

We colour the edges $e_i$; $i = 0, 1, \ldots, 3k-7$ by using the same process given in the result of this lemma. Then we colour the eight edges by *avd-colouring* as

$$\phi(e_i) = \begin{cases} 0, & \text{if } i = 3k-6, 3k-2; \\ 1, & \text{if } i = 3k-5, 3k-1; \\ 2, & \text{if } i = 3k-4, 3k; \\ 3, & \text{if } i = 3k-3, 3k+1. \end{cases}$$

And the colour sets for the vertices $v_i$; $i = 1, 2, \ldots, 3k-4$ are same as the colour sets of the vertices $v_i$; $i = 1, 2, \ldots, 3k-1$ of the above case. For remaining vertices

$$\Phi(v_i) = \begin{cases} [3,0], & \text{if } i = 0, 3k-2; \\ [2,3], & \text{if } i = 3k-3, 3k+1; \\ [0,1], & \text{if } i = 3k-1; \\ [1,2], & \text{if } i = 3k. \end{cases}$$

**Case 4:** If $n = 5$.

Then we colour the edges of $C_5$ as $\phi(e_0) = 0$, $\phi(e_1) = 1$, $\phi(e_2) = 2$, $\phi(e_3) = 3$ and $\phi(e_4) = 4$.

Thus from all above cases we see that minimum colours require to colour the edges of a cycle $C_n$ are 3, when
Proof: Let us consider that the graph $G$ contains two cycles of lengths $n$ and $m$ respectively, joined by a cutvertex $v_0$. Here degree of $v_0$ is 4. Again, let $v_i, e_i : i = 0,1,\ldots,n-1$ and $v_j, e_j : j = 0,1,\ldots,m-1$ be the vertices and edges of $C_n$ and $C_m$ respectively, where $e_0 = (v_0,v_1), e_{n-1} = (v_{n-1},v_0)$ and $e_j = (v_{j+1},v_j)$ for $j = 0,1,\ldots,m-1$. Let $e'_i = (v_{i+1},v_i)$ and $e'_j = (v_{j+1},v_j)$ for $j = 1,2,\ldots,m-2$. Now we colour the edges of the graph by avg-colouring as follows.

**Case 1:** If $n = 3k = 0 \pmod 3$ and $m = 3k = 0 \pmod 3$.

First we colour the edges of $C_n$ as per the rule of avg-edge colouring in case 1 of previous lemma. Now, we colour the edges $e'_i; j = 3,4,\ldots,3k-2$ of $C_n$ as

$$\phi(e_i) = \begin{cases} 1, & \text{if } i = 0 \pmod 3; \\ 2, & \text{if } i = 1 \pmod 3; \\ 0, & \text{if } i = 2 \pmod 3. \end{cases}$$

And the remaining edges as

$$\phi(e'_i) = \begin{cases} 1, & \text{if } j = 0; \\ 2, & \text{if } j = 1; \\ 0, & \text{if } j = 2; \\ 3, & \text{if } j = 3k - 1. \end{cases}$$

**Case 2:** If $n = 3k = 0 \pmod 3$ and $m = 3k + 1 = 1 \pmod 3$.

Here we colour the edges of $C_n$ as same as given in case 1 of previous lemma. Now, we colour the edges $e'_i; j = 3,4,\ldots,3k-2$ of $C_n$ as same process which has done in the above case for the edges $e'_i: j = 3,4,\ldots,3k-2$. Then the colouring procedure for the edges $e'_i; j = 0,1,3,3k$ are

$$\phi(e'_i) = 1, \phi(e'_i) = 2, \phi(e'_i) = 0, \text{ and } \phi(e'_{3k+1}) = 3.$$
\[ e'_0 = (v_0, v_1), e_{m-1}' = (v_{m-1}, v_0) \quad \text{and} \quad e' = (v_j, v_{j+1}) \quad \text{i.e.,} \]
\[ i = 1, 2, \ldots, n-2 \]
\[ e'_0 = (v_0, v_1), e'_1 = (v_1, v_2), \ldots, e'_{m-1} = (v_{m-1}, v_0) \quad \text{and} \]
\[ e'_j = (v_j, v_{j+1}) \quad \text{j.e.,} \]
\[ j = 1, 2, \ldots, m-2 \]
\[ e''_0 = (v_0, v_{p+1}), e''_{m-1} = (v_{p}, v_p') \quad \text{and} \quad e''' = (v_j', v_{j+1}') \quad \text{r.e.,} \]
\[ r = 1, 2, \ldots, p-2. \]

First we colour two cycles according to the rule describe in Lemma 3. Now we colour the edges of \( C \), as follows.

**Case 1:** For \( n = 3k \), \( m = 3k \) and \( p = 3k + i \), \( i = 0, 1, 2 \).

When \( p = 3k \), then we colour edges \( e'' \); \( r = 1, 2, \ldots, 3k - 2 \) by
\[ \phi(e'') = \begin{cases} 
2, & \text{if } r = 0 \pmod{3}; \\
0, & \text{if } r = 1 \pmod{3}; \\
1, & \text{if } r = 2 \pmod{3}.
\end{cases} \]

And the remaining two edges by \( \phi(e''_0) = 4 \) and
\( \phi(e''_{3k-1}) = 5 \).

When \( p = 3k + 1 \), then colouring scheme for the edges \( e''; \ r = 1, 2, \ldots, 3k - 1 \) are
\[ \phi(e'') = \begin{cases} 
2, & \text{if } r = 0 \pmod{3}; \\
0, & \text{if } r = 1 \pmod{3}; \\
1, & \text{if } r = 2 \pmod{3}.
\end{cases} \]

And for the edges \( e''_0 \) and \( e''_{3k-1} \) are \( \phi(e''_0) = 4 \) and
\( \phi(e''_{3k-1}) = 5 \).

When \( p = 3k + 2 \), then we colour the edges \( e'' \), \( e''_{3k+1} \) and \( e'''' \); \( r = 1, 2, \ldots, 3k \) as
\[ \phi(e'') = \begin{cases} 
4, & \text{if } r = 0; \\
5, & \text{if } r = 3k + 1; \\
2, & \text{if } r = 0 \pmod{3}; \\
0, & \text{if } r = 1 \pmod{3}; \\
1, & \text{if } r = 2 \pmod{3},
\end{cases} \]

respectively.

**Case 2:** For \( n = 3k + i \); \( i = 0, 1, 2 \), \( m = 3k + j \) and \( p = 3k + j; \ j = 2 \).

When \( n = 3k + i \); \( i = 0, 1 \), \( m = 3k + 1 \) and \( p = 3k + 1 \), then we colour the edges of \( C_p \) according to the rule of case 1 (for \( 3k + 1 \) of this lemma.

When \( n = 3k + i \); \( i = 0, 1, 2 \), \( m = 3k + j \); \( j = 1, 2 \) and \( p = 3k + 2 \), then the colouring procedure of the edges of \( C_p \) is same as given in case 1 (for \( 3k + 1 \) of this lemma.

**Case 3:** For \( n = 5 \), \( m = 5 \) and \( p = 5 \).

Here we colour the edges \( e'' \); \( r = 0, 1, \ldots, 4 \) as
\[ \phi(e''_0) = 3 \quad \phi(e''_1) = 1 \quad \phi(e''_2) = 0 \quad \phi(e''_3) = 2 \quad \text{and} \quad \phi(e''_4) = 5. \]

So, from the all above cases we see that six colours (which is equal to degree of cutvertex) are needed to colour the edges of the graph.

Thus, \( \chi'_c(G) = 6 (= \Delta) \).

### D. Auv-edge coloring of finite number of cycles

From lemma 3, lemma 4 we can conclude the general form of these lemmas which is given below.

**Lemma 5** If the graph \( G \) contains finite cycles of finite lengths, joined with a common cutvertex with degree \( \Delta \), then \( \chi'_{c}(G) = \Delta \).

**Proof:** From Lemmas 2, 3 and 4, we see that \( \chi'_{c}(C_3) = \Delta + 3 \), \( \chi'_{c}(C_3 \cup C_4) = \Delta + 1 \), where \( C_4 \), \( C_5 \) are two different cycles of lengths 5 joined by \( v_0 \) and \( \chi'_{c}(C_3 \cup C_3 \cup C_5) = \Delta \) respectively, where \( C_4 \), \( C_5 \) and \( C_6 \) are three different cycles of lengths 5. When a graph \( G \) contains two or three cycles of finite lengths other than 5 joined by a common cutvertex, then the value of \( \chi'_{c} \) is \( \Delta \). So, if we prove for a graph which contains finite number of cycles of lengths 5, the value of \( \chi'_{c} \) is equal to degree of the cutvertex, then general result will be proved automatically.

Let \( G \) contains \( k \) number of cycles of lengths 5 (shown in Fig. 2), joined by a common cutvertex \( v_0 \) with degree \( \Delta \). So, \( \Delta = 2k \).

Let \( v_0, v_1, v_2, v_3, v_4, \ldots, v_6, v_{k+1}, \ldots, v_{4k-1}; i = 0, 1, \ldots, 4 \) be the vertices of \( G \). And \( e'; e''; e'''; e^{(k-3)}; i = 0, 1, \ldots, 4 \) are the edges, where \( e'_0 = (v_0, v_1), e_4 = (v_0, v_6) \) and \( e''_i = (v_j, v_{j+1}) \); \( i = 1, 2, 3 \) \( e''''(p) = (v_0, v_1(p)) e''''''(p) = (v_0, v_1(p)) \) and \( e''''''(p) = (v_j(p), v_{j+1}(p)); p = 1, 2, \ldots, k-1; i = 1, 2, 3 \).

![Fig. 2: A graph G contains k number of different C_5's.](image)

Now we colour the edges of first three cycles of lengths 5 by avd-colouring according to the process as given in case 5 of Lemma 4. Then we colour other edges as follows
\[ \phi(e'_i) = \begin{cases} 
2j, & \text{if } i = 0; \\
2j+1, & \text{if } i = 4 \text{ and } j = 3, 4, \ldots, k-1, \\
0, & \text{if } i = 1; \\
1, & \text{if } i = 2; \\
2, & \text{if } i = 3 \text{ and } j = 3, 4, \ldots, k-1.
\end{cases} \]

Here, \( \phi(e''(k-3)) = (2(k-1)+1) = 2k - 1 \).

Thus, \( \chi'_{c}(G) = \Delta \).
IV. AVD-EDGE COLOURING OF OTHER SUBGRAPHS OF CACTUS GRAPH

Lemma 6 Let \( G \) be a graph contains finite number of cycles of finite lengths and finite number of edges, joined with a common cutvertex. If \( \Delta \) be the degree of the cutvertex, then \( \chi'_a(G) = \Delta \).

Proof: According to the previous lemma we know that if a graph contains finite number of cycles of any lengths and they are joined by a common cutvertex with degree \( \Delta \), then \( \chi'_a(G) = \Delta \). So, if we prove that \( \chi'_a \) equal to \( \Delta \) for \( G \), contains \( n \) cycles of lengths \( q \) and \( q \) number of edges, joined by a cutvertex \( v_0 \) with degree \( \Delta \), then we can say that the above statement is true. Here \( \Delta = 2n + q \).

Let \( \epsilon_n^i : i = 0, 1, \ldots, q-1 \) be the edges incident on the common cutvertex \( v_0 \) of \( G \). Now we colour the edges of \( n \) cycles by avd-colouring using the rule given in Lemma 5. Then we colour the \( q \) edges as

\[
\phi(e_0^i) = (2n-1)+(i+1) \quad i = 0, 1, \ldots, q-1.
\]

Here \( \phi(e_0^i) = 2(n-1)+q = 2n+q-1 \). So, \( (2n+q) \) colours are required.

Therefore, \( \chi'_a(G) = \Delta \).

A. Avd-edge coloring of sun

The sun graph is obtained by adding an edge to each of the vertex of any cycle of finite length. If we add an edge to each vertex of \( C_n \), then we get a sun \( S_{2n} \) with \( 2n \) vertices. So \( C_n \) is a subgraph of \( S_{2n} \). We prove the following result for sun.

Lemma 7 For any sun \( S_{2n} \),

\[
\chi'_a(S_{2n}) = \begin{cases} 
\Delta + 2, \text{if } n = 5; \\
\Delta + 1, \text{otherwise}; 
\end{cases}
\]

where \( \Delta = 3 \).

Proof: Let \( v_i, e_i : i = 0, 1, \ldots, n-1 \) be the vertices and edges of the cycle \( C_n \). To construct \( S_{2n} \) (shown in Fig. 3), we add an edge \( e'_i = (v_i, v'_{i+1}) \) to the vertex \( v_i \). The adjacent vertices of \( v_i \) of \( C_n \) are \( v_{i-1} \) and \( v_{i+1} \). And \( v_i' \)'s are all pendent vertices. To colour the edges of sun by avd-colouring, we consider the following four cases.

![Fig. 3: Sun (S_{2n})](image)

Case 1: If \( n \equiv 0 \pmod{3} \), i.e., \( C_{3k} \).

Here we colour the edges of \( C_{3k} \) as per the rule given in case 1 of Lemma 2. And other edges of \( S_{2n} \) as

\[
\phi(e'_i) = \begin{cases} 
1, \quad \text{if } i = 0 \pmod{3}; \\
2, \quad \text{if } i = 1 \pmod{3}; \\
0, \quad \text{if } i = 2 \pmod{3}; 
\end{cases}
\]

Case 2: If \( n \equiv 1 \pmod{3} \), i.e., \( C_{3k+1} \).

The colouring procedure of the edges of \( C_{3k+1} \) are same as given in case 2 of Lemma 2. The colours of the edges \( e'_i \); \( i = 0, 1, \ldots, 3k-1 \) are given by

\[
\phi(e'_i) = \begin{cases} 
1, \quad \text{if } i = 0 \pmod{3}; \\
2, \quad \text{if } i = 1 \pmod{3}; \\
0, \quad \text{if } i = 2 \pmod{3}; 
\end{cases}
\]

and \( \phi(e'_i) = 0 \).

Case 3: If \( n \equiv 2 \pmod{3} \), i.e., \( C_{3k+2} \).

Here we colour the edges of \( C_{3k+2} \) as per the rule given in case 3 of Lemma 2. The edges \( e'_i \); \( i = 0, 1, \ldots, 3k-2 \) of \( S_{2n} \) as

\[
\phi(e'_i) = \begin{cases} 
1, \quad \text{if } i = 0 \pmod{3}; \\
2, \quad \text{if } i = 1 \pmod{3}; \\
0, \quad \text{if } i = 2 \pmod{3}; 
\end{cases}
\]

and \( \phi(e'_i) = 0 \).

Case 4: If \( n = 5 \).

Then we colour the edges \( e'_i \); \( i = 0, 1, 2, 3, 4 \) of \( S_{2n} \) as

\[
\phi(e'_i) = \begin{cases} 
1, \quad \text{if } i = 0; \\
2, \quad \text{if } i = 1; \\
0, \quad \text{if } i = 2, 3, 4. 
\end{cases}
\]

Thus, from all the above cases we find that 4 or 5 colours are required to colour the edges of a sun. For sun \( \Delta = 3 \).

Therefore, \( \chi'_a(S_{2n}) = \begin{cases} 
\Delta + 2, \text{if } n = 5; \\
\Delta + 1, \text{otherwise}; 
\end{cases} \)

Let a graph be obtained from the sun \( S_{2n} \) by joining an edge to each of the pendent vertex. We obtain the following result for such a graph.

Lemma 8 Let \( G \) be a graph obtained from \( S_{2n} \), by adding an edge to each of the pendent vertex, then

\[
\chi'_a(G) = \begin{cases} 
5, \text{if } n = 5; \\
4, \text{otherwise}; 
\end{cases}
\]

The is another important result of avd-colouring of a subgraph of cactus graph which is stated below.

Lemma 9 Let \( G \) be a graph contains a cycle of any length and finite number of edges. If they are joined by a common cutvertex \( v_0 \) with degree \( \Delta \), then \( \chi'_a(G) = \Delta \).

Corollary 1 If the length of the cycle is 5, then,

\[
\chi'_a(G) = \begin{cases} 
\Delta + 2, \text{if one edge incident on cutvertex}; \\
\Delta + 1, \text{if two edges incident on cutvertex}; 
\end{cases}
\]
**Lemma 10** If a graph $G$ contains two cycles of finite lengths and they are joined by an edge, then

$$\chi'_c(G) = \begin{cases} 
\Delta + 2, & \text{if lengths of two cycles are } 5; \\
\Delta + 1, & \text{otherwise}; 
\end{cases}$$

where $\Delta = 3$.

**Proof:** Let $C_n$ and $C_m$ be two cycles joined by an edge $(v_0, v'_0)$ (shown in Fig. 4).

Let $v_i, e_i, j = 0,1,\ldots,m-1$ be the vertices and edges of $C_n$ and $C_m$ respectively.

We colour the edges of $C_n$ as per the rule given in Lemma 2. And we colour the edge $(v_0, v'_0)$ by 1, i.e., $\phi(v_0, v'_0) = 1$. Now the avd-edge colouring procedure of the cycle $C_n$ is given below.

**Case 1:** For $n = 3k, m = 3k + i, i = 0,1,2$.

When $m = 3k$, then we colour the first $3k-1$ edges of $C_n$ as

$$\phi(e_i) = \begin{cases} 
0, & \text{if } i = 0 \text{ (mod 3)}; \\
1, & \text{if } i = 1 \text{ (mod 3)}; \\
2, & \text{if } i = 2 \text{ (mod 3)}; 
\end{cases}$$

and the last edge as $\phi(e_{3k-i}) = 3$.

When $m = 3k+1$ and $3k+2$, the colouring process is same as above.

**Case 2:** For $n = 3k + 1, m = 3k + i, i = 1,2$.

When $m = 3k+1$, then we colour the first $3k-1$ edges of $C_n$ as per second subcase of case 1. We colour the remaining two edges as

$$\phi(e'_i) = \begin{cases} 
3, & \text{if } j = 3k; \\
2, & \text{if } j = 3k + 1. 
\end{cases}$$

When $m = 3k+2$, then we colour the first $3k-1$ edges by using the rule given in above subcase. Now we colour the last three edges as

$$\phi(e'_i) = \begin{cases} 
3, & \text{if } j = 3k; \\
0, & \text{if } j = 3k + 1; \\
2, & \text{if } j = 3k + 2. 
\end{cases}$$

**Case 3:** For $n = 3k + 2, m = 3k + 2$.

Here we colour the edges of $C_n$ according to the rule of second subcase of case 2.

When $n = 5$ and $m = 5$, then we colour the edges as

$$\phi(e_i) = \begin{cases} 
0, & \text{if } j = 0 \text{ (mod 3)}; \\
1, & \text{if } j = 1 \text{ (mod 3)}; \\
2, & \text{if } j = 2 \text{ (mod 3)}. 
\end{cases}$$

So, from the above cases we see that $\chi'_c$ equal to 5 when two cycles are of lengths 5, and 4 for other cases. In this case $\Delta = 3$.

Therefore,

$$\chi'_c(G) = \begin{cases} 
\Delta + 2, & \text{if lengths of two cycles are } 5; \\
\Delta + 1, & \text{otherwise}; 
\end{cases}$$

where $\Delta = 3$ is the degree of $G$.

**Lemma 11** If a graph $G$ contains two suns and they are joined by only one edge, then

$$\chi'_c(G) = \begin{cases} 
\Delta + 2, & \text{if lengths of two cycles are } 5; \\
\Delta + 1, & \text{otherwise}; 
\end{cases}$$

**Proof:** Let $v_0, v_1, \ldots, v_n, e_0, e_1, \ldots, e_{n-1}$ be the vertices and edges of $P_n$. When $n = 2,3$ and 3 colours when $n > 3$.

So, we need 1 and 2 colours when $n = 2,3$ and 3 colours when $n > 3$.

Therefore, $\chi'_c(P_n) = \begin{cases} 
1, & \text{if } n = 2,3; \\
3, & \text{if } n > 3. 
\end{cases}$

**V. AVD-EDGE COLOURING OF CATERPILLAR**

**Definition 1** A caterpillar $C$ is a tree where all vertices of degree $\ge 3$ lie on a path, called the backbone of $C$. The hairlength of a caterpillar graph $C$ is the maximum distance of a non-backbone vertex to the backbone.

The result for any caterpillar graph is give below.

**Lemma 14** For any caterpillar graph $G$, 

$$\phi(e'_i) = \begin{cases} 
0, & \text{if } j = 0 \text{ (mod 3)}; \\
1, & \text{if } j = 1 \text{ (mod 3)}; \\
2, & \text{if } j = 2 \text{ (mod 3)}. 
\end{cases}$$
\[
\chi'_v(G) = \begin{cases} 
\Delta, & \text{if two vertices of maximum degree are not adjacent;} \\
\Delta + 1, & \text{if two vertices of maximum degree are adjacent.}
\end{cases}
\]

**Proof:** Let the length of the path of the caterpillar be \( n \). Then we colour the edges of the path by \( \text{avd-colouring} \) using Lemma 13.

Here only we prove \( \chi'_v(G) = \Delta + 1 \), when two vertices with maximum degree (\( \Delta \)) are adjacent. The graph of Fig. 5 is a caterpillar. Let \( u \), \( v \) be the vertices with maximum degree of \( G \) and they are adjacent. That is, \( d(u) = d(v) = \Delta \).

Again let \( u_i \)’s and \( v_i \)’s be the adjacent vertices of \( u \) and \( v \) respectively, where \( i = 1, 2, \ldots, \Delta - 1 \).

Without loss of generality let the colour of the edge \((u, v)\) be 0.

![Fig. 5: A caterpillar](image)

Then \( \phi(u, v) = 0 \), \( \phi(u_i, u_j) = i \) (\( i = 1, 2, \ldots, d(u) - 1 \)), \( \phi(v, v_i) = i + 1 \) (\( i = 1, 2, \ldots, d(v) - 1 \)).

So, there are \( 0, 1, 2, \ldots, \Delta \), i.e., total \( \Delta + 1 \) colours are required.

In the caterpillar all the vertices degree \( \geq 3 \) must lie on the path. So the vertices which are at a distance 2, 3, 4, \ldots, from the path are of degree either 1 or 2. Let \( u'_1, u''_1, u'''_1, \ldots, \) be the vertices at distance 2, 3, 4, \ldots, \). So we colour the edges \((u'_1, u'_2), (u''_1, u''_2), (u'''_1, u'''_2), \ldots, \) from any of the colour \( 0, 1, 2, \ldots, \Delta \), such a way that no two adjacent vertices have the same set of colours. The rule is similar for the edges \((v'_1, v'_2), (v''_1, v''_2), (v'''_1, v'''_2), \ldots, \).

**VI. AVD-EDGE COLOURING OF LOBSTER**

Another subclass of cactus graph is called lobster graph. The definition of lobster graph is given below.

**Definition 2** A lobster is a tree having a path (of maximum length) from which every vertex has distance at most \( k \), where \( k \) is an integer.

The maximum distance of the vertex from the path is called the diameter of the lobster graph. There are many types of lobsters given in literature like diameter 2, diameter 4, diameter 5, etc.

From definition of lobster we know that lobster is a one kind of tree. So, by using the result of \( \text{avd-edge colouring} \) for any tree [15], we can conclude the result for lobster which is stated below.

**Lemma 15** For any lobster \( G \),

\[
\chi'_v(G) = \begin{cases} 
\Delta, & \text{if two vertices of maximum degree are not adjacent;} \\
\Delta + 1, & \text{if two vertices of maximum degree are adjacent.}
\end{cases}
\]

**Lemma 16** Let \( G_1 \) and \( G_2 \) be two cactus graphs. If \( \Delta_1 \leq \chi'_v(G_1) \leq \Delta_1 + 3 \) and \( \Delta_2 \leq \chi'_v(G_2) \leq \Delta_2 + 3 \), then, \( \Delta \leq \chi'_v(G) \leq \Delta + 3 \), where \( G = G_1 \cup G_2 \).

**Proof:** Let \( G_1 \) and \( G_2 \) be two cactus graphs and \( \Delta_1 \), \( \Delta_2 \) be the degrees of them. Now if we merge two cactus graphs with the vertex \( v \) then we get a new cactus graph \( G \) (\( = G_1 \cup G_2 \)). Let \( \Delta \) be the degree of \( G \) and we will prove that max \( \{\Delta_1, \Delta_2 \} \leq \Delta_1 + 3 \) . For the graph \( G_1 \), \( \Delta_1 \leq \chi'_v(G_1) \leq \Delta_1 + 3 \) and for the graph \( G_2 \), \( \Delta_2 \leq \chi'_v(G_2) \leq \Delta_2 + 3 \). Here we have to prove that the lower and upper bounds will preserve for the new graph \( G \).

Let maximum degree \( \Delta_1 \) of \( G_1 \) be attained at \( u \) and \( v \) be a vertex of \( G_2 \) whose degree is \( \Delta_2 \). Again, let \( u_i \)’s and \( v_i \)’s be the adjacent vertices of \( u \) and \( v \) respectively, where \( i = 1, 2, \ldots, \Delta_1 \), \( j = 1, 2, \ldots, \Delta_2 \). We merge \( G_1 \) and \( G_2 \) at \( v \) and let the adjacent vertices of \( v \) be \( v_i \); \( i = 1, 2, \ldots, \Delta_1 + \Delta_2 \). If we merge one vertex of \( G_1 \) other than \( u \) with \( v \) of \( G_2 \) then the adjacent vertices of \( v \) are \( v_i \); \( i = 1, 2, \ldots, \max \{\Delta_1, \Delta_2 \} \). So we can say that \( \Delta \) lies between max \( \{\Delta_1, \Delta_2 \} \) and \( \Delta_1 + \Delta_2 \).

Now from this result we can conclude that the upper and lower bounds of \( \chi'_v \) remains same as \( G \).

The \( \text{avd-edge colouring} \) of all subgraphs of cactus graphs and their combinations are discussed in the previous lemmas. From these results we conclude that \( \chi'_v \) value of any cactus graph can not be more than \( \Delta + 3 \) . Hence we have the following theorem follows.

**Theorem 1** If \( \Delta \) is the degree of a cactus graph \( G \), then \( \Delta \leq \chi'_v(G) \leq \Delta + 3 \).

**Proof.** The \( \text{avd-edge colouring} \) of all possible subgraphs of cactus graph are discussed and have shown that \( \Delta \leq \chi'_v(G) \leq \Delta + 3 \). Let \( G \) be obtained by \( v \)-union of two cactus graphs then \( G \) becomes a cactus graph and it is proved that \( \chi'_v(G) \) should satisfy the inequality \( \Delta \leq \chi'_v(G) \leq \Delta + 3 \). (Lemma 16).

Hence the theorem.
VII. THE ALGORITHM AND ITS TIME COMPLEXITY

To start the algorithm of avd-edge colouring of a cactus graph, we first construct a graph \( G' \) which is equivalent to the given graph \( G \).

A. Construction of an equivalent graph \( G' \) of \( G \)

Using DFS we obtain all blocks and cutvertices of a cactus graph \( G = (V, E) \). Let the blocks be \( B_0, B_1, B_2, ..., B_{k-1} \) and the cutvertices be \( C_1, C_2, C_3, ..., C_R \) where \( N \) is the total number of blocks and \( R \) is the total number of cutvertices.

The blocks of the cactus graph shown in Fig. 6 are

\[
\begin{align*}
B_0 &= (7, 8, 9, 10, 11, 12, 13),
B_1 &= (11, 14, 15, 16, 17, 18, 19),
B_2 &= (12, 19, 20, 21),
B_3 &= (7, 38),
B_4 &= (7, 39),
B_5 &= (7, 1),
B_6 &= (16, 20),
B_7 &= (16, 23, 24, 25),
B_8 &= (16, 26),
B_9 &= (1, 40),
B_{10} &= (1, 2, 3, 4, 5, 6),
B_{11} &= (20, 21),
B_{12} &= (20, 22),
B_{13} &= (26, 27),
B_{14} &= (3, 41),
B_{15} &= (4, 42, 43).
\end{align*}
\]

Fig. 6: A cactus graph \( G \)

and the cutvertices are \{7, 1, 4, 3, 9, 11, 12, 16, 18, 20, 26\} respectively.

Now we have in a position to construct an equivalent graph \( G' \) of \( G \) whose vertices are the blocks of \( G \) and an edge is defined between two blocks if they are adjacent blocks of \( G \), i.e., \( G' = (V', E') \) where \( V' = \{B_0, B_1, ..., B_{k-1}\} \) and \( E' = \{(B_i, B_j) : i \neq j, i, j = 0, 1, ..., N-1 \} \).

B. Avd-colouring of edges

Now, we take any arbitrary cycle as starting block. The block is so chosen that the degree of the cutvertex is maximum. We denote the starting block as \( B_0 \) and let the level of \( B_0 \) be 0. Now, the blocks adjacent to \( B_0 \) are taken at level 1. The blocks adjacent to the blocks of level 1 are taken as the blocks of level 2 and so on. Now we colour the block \( B_0 \) using the rule stated at Lemma 2. Now we colour the blocks of level 1 from left to right. Let the blocks of level 1 be \( B_{11}, B_{12}, B_{13}, ... \). They are either edges or cycles. We consider the first block \( B_{11} \) of level 1 which is adjacent to \( B_0 \). If the block is an edge then colour the block by using Lemma 7. If it is a cycle of finite length then we colour it by using Lemma 3.

The blocks which are adjacent to \( B_0 \) we colour them according to the rule of the block \( B_{11} \). Next we colour the edges of the block \( B_{12} \). Suppose it is not adjacent to \( B_{11} \). Then if \( B_{12} \) is an edge, we use Lemma 7 to colour the edge. If \( B_{12} \) is a cycle, then we colour it by using Lemma 8.

\textbf{Algorithm MINAVDEC}

\textbf{Input:} The cactus graph \( G = (V, E) \).

\textbf{Output:} Avd-colouring of its edges.

Step 1: Compute the blocks and cutvertices of \( G \) and
construct an equivalent graph $G'$ of $G$.

Step 2: Let $B_0$, starting block, where the degree of the cutvertex of $B_0$ is maximum.

Step 3: We colour the block $B_0$ using Lemma 2.

Step 4: Consider the blocks $B_{j,i}$, $j=1, 2, 3, \ldots$, of level 1. Colour the blocks from left to right as follows.

(i) Take the first block $B_{11}$ which is adjacent to $B_0$. If it is an edge then we colour $B_{11}$ by using Lemma 7 and if it is cycle then we colour it by Lemma 3.

(ii) Next we consider the second block $B_{12}$. If it is not adjacent to $B_{11}$, then colour it by either Lemma 7 or Lemma 3. If $B_{12}$ adjacent to $B_{11}$ then colour it by using lemmas 5, 6, 9 and 16.

(iii) Consider the block $B_{13}$. If it is not adjacent to any block of level 1, then colour it by lemma 7 or 3. But if it is adjacent to at least one block of level 1 then follow the rules of lemmas 5, 6, 9 and 16.

(iv) The blocks which are adjacent to $B_0$ only then colour them by the process similar to $B_{11}$.

Step 5: Suppose a block at level $l$, say $B_{l,0}$ is an edge and another block $B_{l+1,i}$ at level $l+1$ adjacent to $B_{l,0}$, which is also an edge. Then we colour them by using Lemma 8.

Step 6: Let a block $B_{l-1,j}$ at level $k-1$ be a cycle length and its adjacent block at level $k$ be $B_{l,i}$ is an edge. If its adjacent block $B_{l+1,j}$ is a cycle of finite length at level $k+1$, then colour the blocks by using Lemma 10.

Step 7: Suppose $B_{q-2,i}$ is a block at level $q-2$ which is a cycle of finite length. Its adjacent block $B_{q-1,i}$ at level $q-1$ is an edge. The adjacent block of $B_{q-1,j}$ at level $q$ is $B_{q,m}$, cycle of finite length. Let every vertex $B_{q,m}$ contains an edge, if $B_{q+1,n}$ one of them at level $q+1$, then we colour the blocks by using Lemma 11.

Step 8: Consider the blocks of subsequent levels and repeat steps 4 to step 7 to colour all the vertices of $G$.

end MINAVDEC

C. Time Complexity

The correctness of the algorithm follows from the lemmas proved in the paper.

Theorem 2 The time complexity of the algorithm MINAVDEC is $O(n)$.

Proof. The blocks and cutvertices of any graph can be computed in $O(m+n)$ time [14]. For the cactus graph $m = O(n)$. Hence step 1 of algorithm MINAVDEC takes $O(n)$ time.

The time complexity to colour the edges of a block of size $m_i$ is $O(m_i)$. Step 4 colours the vertices of the blocks which are at level 1 of $G'$. If the number of vertices of all blocks of this level is $m_2$, then the time complexity for step 4 is $O(m_2)$. That is the time complexity depends upon the number of vertices of the whole graph. Since the number of vertices of the entire graph is $n$, the time complexity of the algorithm is $O(n)$.

VIII. CONCLUSION

The bounds of avd-edge colouring of a cactus graph and various subclass viz., cycle, sun, star, caterpillar, lobster are investigated. By merging all the results, we have observed that for any cactus graph the value of avd-edge chromatic number lies between $\Delta$ and $\Delta + 3$. Currently we are engaged to find the bounds for different graph labelling like list coloring, graceful labeling, Harmonious labeling, etc, on cactus graphs.

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Dr. Nasreen Khan is Assistant Professor of Mathematics Department, Global Institute of Management and Technology, West Bengal, India. Before that she was a fulltime research scholar of Applied Mathematics Department, Vidyasagar University. She has completed her Phd in 2013 under the guidance of Professor Madhumangal Pal, the Professor of Applied Mathematics Department, Vidyasagar University. Her B.Sc and M.Sc both were from Vidyasagar University with good marks. Dr. Khan has six articles of which two are in national and four in international journals. Her specialization is Computational Graph Theory. Her interest for future research is in Fuzzy Game Theory, Fuzzy Graph Theory, Optimization.

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