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A New Method for Solving Bi Criterion Linear Fractional Programming Problems

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Abstract -In this paper we present a new feasible direction method to find all efficient extreme points for bi criterion linear fractional programming problems. This method is based on the conjugate gradient projection method. An initial feasible point is used to generate all efficient extreme points for this problem through a sequence of feasible directions of movement. Since methods based on vertex information may have difficulties as the problem size increases we expect this method to be less sensitive to problem size. A simple example is given to clarify the theory of this new method.

Keywords: **Multi criterion linear fractional programming, bi criterion fractional programming efficient solution, non-dominated solution.**

I. INTRODUCTION

Multi - criterion linear fractional programming (MCLFP) problems have attracted researches in the last two decades. This kind of problems arises when several linear fractional functions (i.e. ratio objectives that have linear numerator and denominator) are to be maximized over a convex constraints polytope X. The (MCLFP) problems are useful in production planning, financial planning and corporate planning, health care, and hospital planning. Few approaches have been reported for solving (MCLFP). Kornbluth and Steuer (1981) considered this problem and presented a simplex –based solution procedure to find all weakly efficient vertices of the augmented feasible region. Benson (1985) in his article showed that the procedure suggested by Kornbluth and Steuer for computing the numbers to find break points may not work all the time and he proposed a failsafe method for computing these numbers. A survey note appeared in [18] about the methods used for solving the linear fractional programming problems with several objective functions. Also in [16, 17], amount of work which has been done since 1980, on solving the problem of fractional mathematical programming, in particular, the case of bi criterion which has received a considerable attention. Evidently, any method and result referring to the fractional programming can be applied to the case of two objective functions; however the reduce number of functions offers some advantages; for the case of two objective functions many papers appeared which take the advantage of the particular structure of the problem, see also [9, 12,13,18, 19]. Also multi -criterion linear programming (MCLP) problems with several linear objective functions can be considered as a special case of (MCLFP). Several approaches have been suggested for solving this problem, among them are the ones suggested by [3, 4, 5, 6,10] . Most of these methods depend mainly on the canonical simplex tableau in multiple objective forms to find the efficient set of this problem. More recent work about general multi-criterion mathematical programming problem can be found see [20] for more detail about solving these problems the reader should refer to [2]. In this paper we present a new method to solve the bi criterion linear fractional programming problems This method provides us with a feasible directions of movement from an efficient extreme point to its adjacent one and is based on the conjugate gradient projection method, starting with an initial efficient extreme point we generate a sequence of feasible directions towards all efficient adjacent extremes of the above problem. The structure of the paper is organized as follows: in Section 2 some definitions and notation are given, in Section 3 we gave the proposed algorithm to find the efficient extreme points for bi criterion linear fractional programming problems, and finally a conclusion remarks on solving this kind of problem is given in Section 4

II. DEFINITIONS AND NOTATION

Bi criterion linear fractional programming arises when bi objective linear fractional functions (i.e. ratio objectives that have linear numerator and denominator) are to be maximized over a convex constraints polytope X, this problem can be formulated as

Maximize $Z(x) = (z_1(x), z_2(x))$

Subject to

$x \in X = \{x \in R^n, Ax \le b\}$ (1)

where $$z_i(x) = \frac{c_i^T x + \alpha_i}{d_i^T x + \beta_i}, \quad i = 1, 2$$

Here c_i, d_i are vectors in R^n, α_i and β_i are scalar, A is an (m +n) x n matrix and $b \in R^{m+n}$.

We point out that the non negativity condition is added to the set of constraints and we also assume that X is compact set and $d_i^T x + \beta_i > 0$, i = 1, 2 for every $x \in X$. Solving bi criterion linear fractional programming seeks for the set of solutions called the efficient set every member of this set has the following definition

Definition 2-1:

A solution x^0 is said to be efficient for multiple objective linear fractional programming (MOLFP) if $x^0 \in X$ and these is no $x \in X$ such that $Z(x) \ge Z(x^0)$ and $Z(x) \ne Z(x^0)$. The resulting criterion vector $Z(x^0)$ is said to be non-dominated.

Consider the bi criteria linear programming problem in the form:

Maximize $\Phi(x) = (\Phi_1(x), \Phi_2(x))$

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Subject to

$x \in X = \{x \in R^n, \quad Ax \le b\}$ (2)

where $\Phi_i = (c_i^T - z_i^* d_i^T)x$, i = 1, 2

In (2) $z_i^* = z_i(x^*)$ is a given constant vector computed at a given feasible point x* of the compact set X. Since the level curves of each objective for (1) can be written as

$$(c_i^T - z_i^* d_i^T)x = \beta_i z_i^* - \alpha_i, \qquad i = 1, 2$$

,

then we can find a one to one correspondence between the bi criterion linear fractional programming defined by (1) and the bi criterion linear programming problem as defined in (2) and we have the following proposition

Proposition 2-1

if x^0 solves the bi criterion linear fractional programming problem (1) with objective function values z_i^0 then x^0 solves the bi criterion linear programming problem defined by (2) with objective function value

$$\Phi_i^0 = \beta_i z_i^0 - \alpha_i \qquad , i = 1, 2$$

Proof: straight forward.

Now rewrite the bi criterion linear programming problem (2) in the form

Maximize $Z(x) = Cx$

Subject to

$x \in X = \{x \in R^n, \quad Ax \le b\}$ (3)

where C is a 2 x n matrix whose rows are those represented by $(c_i^T - z_i^0 d_i^T)$, i = 1, 2

The above problem can be considered as an equivalent transformation for problem (1) locally defined at a given point hence due to the well known theorem for general multi-criterion linear programming problems [6, 22] that a feasible point x is efficient solution for (3) if and only if there is $\lambda \in R^2$, $\lambda > 0$ (weights) such that x is the optimal solution for the linear program

Maximize $\lambda^T C x$

Subject to

$Ax \le b$ (4)

Consider the dual problem of the above linear program problem (4) in the form

Minimize $u^T b$

Subject to

$u^T A = \lambda^T C$ (5)

$u \ge 0$

Since the set of constraints of this dual problem is written in matrix form then we can multiply both sides by a matrix $T = (T_1 \mid T_2)$, where $T_1 = C^T (C C^T)^{-1}$, and the column of the matrix T_2 constitute the bases of $N(C) = \{v;\ C v = 0\}$,

then we have $u^T A T_1 = \lambda^T$, $u^T A T_2 = 0$ and $u \ge 0$.

If n =2, we have $AT_2 = 0$, the dual of (5) takes the form

Maximize $\lambda^T y$

Subject to

$A T_1 y \le b$ (6)

where T_1 is the inverse matrix of the given matrix C. On the other hand if $AT_2 \ne 0$. an L x (m + n) matrix G of non-negative entries is defined such that $GAT_2 = 0$, this matrix can be used for the construction of the set of objective space Y in the form

$Y = \{y \in R^2 \mid GAT_1 y \le G b\}$

which can be written in a simple form as

$Y = \{y \in R^2 \mid Q y \le q\}$,

where $Q = GAT_1$ and $q = G b$, the general case the dual of (5) can take the form

Maximize $\lambda^T y$

Subject to

$Q y \le q$ (7)

Remark **2-1**: The matrix G of non-negative entries such that $GAT_2 = 0$, can be considered as a polar matrix of the given matrix AT_2 , also a sub matrix $\bar{G}$ of G satisfying $\bar{G} AT_1 = \lambda^T > 0$ can play an important rule for specifying the positive weights needed for detecting the non dominated point for this bi criterion linear program. In what follows we shall give an iterative method to find the first efficient extreme point for the bi criterion linear programming problem (3). Consider the linear programming problem

Maximize $F(x) = e^T Cx$

Subject to:

$Ax \le b$

where $e \in R^2$ with all entries equals 1. If $e^T C = p^T$, then the above linear program can be written as

Maximize $F(x) = p^T x$

Subject to

$Ax \le b$ (8)

The above linear programming problem can also be written as

Maximize $F(x) = p^T x$

Subject to (9)

$a_\delta^T x \le b_\delta$, $\delta = 1, 2\dots, m + n$.

where a_δ^T represents the δ th row of the given matrix A , then we have in the non degenerate case an extreme point (vertex) of x is defined on some n linearly independent subset of X .Now our task is to find the optimal extreme point for the above linear

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programming problem (first efficient extreme point). Starting with an initial feasible point we shall generate a sequence of feasible directions toward optimality, in general if x^{k-1} is a feasible point obtained at iteration k-1 (k = 1, 2 …) then at iteration k we can find a new feasible point x^k given by

$$x^k = x^{k-1} + \gamma_{k-1}\mu^{k-1} \quad (10)$$

where μ^{k-1} is the direction vector along which we move and given by

$$\mu^{k-1} = H_{k-1}\, p \quad (11)$$

Here H_{k-1} is an n x n symmetric matrix given by

$$H_{k-1} = \begin{cases} I & \text{for } k=1 \\ H^q_{k-1} & \text{If } k>1 \end{cases}$$

while H^q_{k-1} is defined as follows, for each active constraint s; s = 1, 2…, q.
at the current point

$$H^q_{k-1} = H^{s-1}_{k-1} - \frac{H^{s-1}_{k-1} a_s a_s^T H^{s-1}_{k-1}}{a_s^T H^{s-1}_{k-1} a_s} \quad (12)$$

with H^0_{k-1} = I .Then H_{k-1} is given by $H_{k-1} = H^q_{k-1}$.
The step length γ_{k-1} is given by

$$\gamma_{k-1} = \min_{\delta=1,\dots,m+n} \{ g_\delta \ / \ g_\delta = \frac{b_\delta - a_\delta^T x^{k-1}}{a_\delta^T \mu^{k-1}},\ \text{and } g_\delta > 0 \} \quad (13)$$

This relation states that γ_{k-1} is always positive.. Also due to the well known Kuhn-Tucker condition [7, 8] for the point x^k to be an optimal solution of the linear program (8) there must exist $u \geq 0$ such that $A_r^T u = p$,

which can be written as $u = A_r (A_r A_r^T)^{-1} p$ (14)

Here A_r is a sub -matrix of the given matrix A containing only the coefficients of the set of active constraints at the current point x^k. This fact will act as a stopping rule when the optimal solution is obtained. A full description of an algorithm for solving (9) can be found in [23] Based on the above results we shall give in the next section our algorithm for solving the equivalent bi criterion linear programming problem (5) to find all efficient extreme points of (1) in two phases as follows:

3- New algorithm for solving bi criteria linear fractional programming problems

Phase I: Use first the linear programming (9) to find an initial efficient extreme point for the equivalent problem defined by (3) through the following steps:

Step 0: set k=1, Ho =I, μ^0 =p, let x^0 be an initial feasible point, and use relation (13) to compute γ_0.
Step 1: Apply relation (10) to find a new solution x^k.
Step 2: Apply relation (14) to compute u , if $u \geq 0$ stop. The current solution x^k is the optimal solution otherwise go to step 3.
Step 3: Set k = k+1, apply relations (12),(11) and (13) to compute H_{k-1}, μ^{k-1} and γ_{k-1} respectively and go to step 1.

In our analysis to find all efficient extreme points we do this by proceeding from an efficient point given through phase I to its adjacent efficient extremes . By defining a frame for Cone (H) denoted by F, called a minimal spanning system. For a n x n matrix H denote the set of indices of the columns of H by Id_H .
Hence if H = $(h^1,\dots,h^n)$, then **I** $d_H=\{1,2\dots n\}$. For a matrix H we define the positive cone spanned by the columns of H (called a conical or positive hull by Stoer and Witzgall [14]) as Cone (H) = Cone (h^i; i ∈ **I** d_H)

$$= \{h \in R^n;\ h = \sum_{i \varepsilon Id_H} \tau_i h^i,\ \tau_i \geq 0\}$$

A frame F of Cone (H) is a collection of columns of H such that Cone (h^i; i ∈ **I** d_H) = Cone (H) and for each j ∈ Id_H , we have

Cone (h^i ;i ∈ **I** $d_H\backslash\{j\}$) ≠ Cone (H)

Based on the above definitions, we start phase II to find all efficient extreme points for the equivalent bi criteria problem as follows

Phase II:
Step 1: Let x^k be an efficient extreme point, compute H_k correspond to this point x^k
Step 2: Construct a frame F of cone H_k using e.g. .the method of Wets and Witzgall [21].
Step 3: for each h^i ∈ F determines t* obtained by solving the system of linear inequalities of the form
$t A h^i \leq b - A x^k$,
(the boundary points of this interval gives t *).
Step 4: compute $\mathbf{x}^{\sigma} = x^k + t^* h^i$**,** as an efficient extreme point for this bi criteria problem, and go to step 1.

An illustrative example
Consider the following bi criterion linear fractional programming problem

$$\text{Maximize } z_1 = \frac{x_1 + x_2 + 2}{x_1 + 1}$$

$$\text{Maximize } z_2 = \frac{2x_1 + 3x_2 + 1}{x_2 + 1}$$

Subject to:
$x_1 + x_2 \leq 4$
$-x_1 + 2x_2 \leq 2$
$x_1 \geq 0, x_2 \geq 0$

For this problem we have $c_1^T = (1\ \ 1)$, $c_2^T = (2\ \ 3)$, $d_1^T = (1\ \ 0)$, $d_2^T = (0\ \ 1)$ also $\alpha_1 = 2$, $\alpha_2 = 1$, $\beta_1 = 1$ and $\beta_2 = 1$, let $x^{T0} = (2\ \ 1)$ be an initial feasible

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point

Then the equivalent bi criterion linear programming problem at this point takes the form

Maximize $z1(x) = -2/3\, x_1 + x_2$

Maximize $z_2(x) = 2x_1 - x_2$
Subject to
$x_1 + x_2 \leq 4$
$-x_1 + 2x_2 \leq 2$ $-x_1 + 2x_2 \leq 2$
$x_1 \geq 0, x_2 \geq 0$

The first efficient extreme point is obtained by solving the (LP) problem

Maximize $z = 4/3\, x_1$
Subject to:
$x_1 + x_2 \leq 4$
$-x_1 + 2x_2 \leq 2$
$x_1 \geq 0, x_2 \geq 0$,

Step 0: k=1, $H_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ $\mu^0 = \begin{pmatrix} 4/3 \\ 0 \end{pmatrix}$, and at the initial point

$x^{T\,0} = (2 \quad 1)$, relation (2.18) gives $\gamma_0 = 3/4$, we go to step 1

Step 1: apply relation (2.14)

to get $x^1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} + 3/4 \begin{pmatrix} 4/3 \\ 0 \end{pmatrix} = \begin{pmatrix} 3 \\ 1 \end{pmatrix}$

Step 2: for this point x^1 the first constraint is the only active constraint and since relation (2.19) is not satisfied this indicates that this point is not optimal, we go to step3.

Step 3: set k = 2, compute

$H_1 = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$, $\mu^1 = \begin{pmatrix} 3/8 \\ -3/8 \end{pmatrix}$ and $\gamma_1 = 8/3$

and we go to step 1, to get

$x^2 = \begin{pmatrix} 3 \\ 1 \end{pmatrix} + 8/3 \begin{pmatrix} 3/8 \\ -3/8 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$

For this point the first and the second constraints are the only active constraints, since (2.19) is satisfied with u =[4/3 4/3] > 0 this indicating that this point x^2 is an optimal for this linear programming and consequently it is the first efficient extreme point generated for this bi criteria linear programming problem and we start phase
Phase II:

At this point x^2 the bi criterion linear program takes the form

Maximize $z_1(x) = -1/5x_1 + x_2$

Maximize $z_2(x) = 2x_1 - 6\, x_2$
Subject to
$x_1 + x_2 \leq 4$
$-x_1 + 2x_2 \leq 2$
$x_1 \geq 0, x_2 \geq 0$

$T_1 = \begin{pmatrix} 30/4 & 5/4 \\ 10/4 & 1/4 \end{pmatrix}$, $T_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$,

hence the matrix P in this case is the identity

matrix and the computed matrix PAT_1 will take the form

$PAT_1 = \begin{pmatrix} 10 & 3/2 \\ -10/4 & -3/4 \\ -30/4 & -5/4 \\ -10/4 & -1/4 \end{pmatrix}$

we have $\lambda^T = [10 \quad 3/2]$ is the only positive weight defined at this point to explore adjacent non dominated points in objective space,a subset of the set of the active constraints at x^2 (the first constraint) is chosen to compute

$H_2 = \begin{pmatrix} 1/2 & -1/2 \\ -1/2 & 1/2 \end{pmatrix}$

A frame of the columns of H_2 is used as feasible directions to find the adjacent extreme point for x^2 by solving the system of linear iniquities

$\begin{pmatrix} 1 & 1 \\ -1 & 2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} \gamma \leq \begin{pmatrix} 0 \\ 6 \\ 4 \\ 0 \end{pmatrix}$

then with $\gamma^* = -4$, we have an adjacent efficient extreme point

$x_1^* = \begin{pmatrix} 4 \\ 0 \end{pmatrix} - 4 \begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

Is explored, and again we repeat the same steps at this point with the bi criteria linear program

Maximize $z_1(x) = -x_1 + x_2$

Maximize $z_2(x) = 2x_1 - 2/3\, x_2$
Subject to
$x_1 + x_2 \leq 4$
$-x_1 + 2x_2 \leq 2$
$x_1 \geq 0, x_2 \geq 0$

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for the above problem we have

$$T_1 = \begin{pmatrix} 1/2 & 3/4 \\ 3/2 & 3/4 \end{pmatrix}, \quad T_2 = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$, and the computed matrix

PAT_1 takes the form

$$PAT_1 = \begin{pmatrix} 2 & 3/2 \\ 5/2 & 3/4 \\ -1/2 & -3/2 \\ -3/2 & -3/4 \end{pmatrix}$$

We note that $\lambda^T = [2 \quad 3/2]$ and $\lambda^T = [5/2 \quad 3/4]$ are the only positive weight defined at this point to explore adjacent non dominated points in objective space

a subset of the set of the active constraints at x_1* (the first constraint) will lead us back to explore the point x^2 again through

$$x_2* = \begin{pmatrix} 2 \\ 2 \end{pmatrix} + 4\begin{pmatrix} 1/2 \\ -1/2 \end{pmatrix} = \begin{pmatrix} 4 \\ 0 \end{pmatrix}$$

On the other hand if we chose the second active constraint to compute

$$H_3 = \begin{pmatrix} 4/5 & 2/5 \\ 2/5 & 1/5 \end{pmatrix}$$

A frame of the columns of H_3 is used as feasible directions to find the adjacent extreme point for x^3 by solving the system of linear iniquities

$$\begin{pmatrix} 1 & 1 \\ -1 & 2 \\ -1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 2/5 \\ 1/5 \end{pmatrix} \gamma \leq \begin{pmatrix} 0 \\ 0 \\ 2 \\ 2 \end{pmatrix},$$

then with $\gamma * = -5$, we have an adjacent efficient extreme point

$$x_3* = \begin{pmatrix} 2 \\ 2 \end{pmatrix} - 5\begin{pmatrix} 2/5 \\ 1/5 \end{pmatrix} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

Is explored and if we repeat the same steps at this point we obtain again the point x_1* , hence the three efficient extreme points x^2 , x_1* and x_3* are the only efficient extreme points for this bi criterion linear fractional programming problem

III. CONCLUSION

This paper presents a new method for solving bi criterion linear fractional programming problems. The proposed method is based on conjugate gradient projection method. The main idea behind our work is to move in the interior of the feasible region through a sequence of points and hence we can consider this work as an interior point method which proved to be less sensitive to problem size compared with methods based on vertex information which have difficulties as the problem size increases.

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