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Fuzzy Symmetric Solutions of Semi-Fuzzy Sylvester Matrix Systems

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***Abstract*—In this paper the fuzzy symmetric solutions of fuzzy matrix equations AX+XB=C in which A and B are crisp matrices and C is an arbitrary fuzzy numbers matrix, is investigated. The fuzzy linear matrix equation is converted into a fuzzy linear system by the Kronecker product of matrices. From solving the fuzzy linear systems, three types of fuzzy symmetric solutions of the fuzzy matrix systems are derived. Finally, an example is given to illustrate the proposed method. Our results enrich the fuzzy linear systems theory.**

***Index Terms*—fuzzy numbers, matrix analysis, fuzzy matrix equations, fuzzy approximate solutions.**

I. INTRODUCTION

System of simultaneous matrix equations is an essential mathematical tool in science and technology. In many applications, at least some of the parameters of the system are represented by fuzzy rather than crisp numbers. So, it is very important to develop a numerical procedure that would appropriately handle fuzzy matrix systems and solve them. The concept of fuzzy numbers and arithmetic operations were first introduced and investigated by Zadeh ***[1]***, Dubois et al. ***[2]*** and Nahmias ***[3]***. A different approach to fuzzy numbers and the structure of fuzzy number spaces was given by Puri and Ralescu ***[4]***, Goetschell et al.***[5]*** and Wu et al. ***[6, 7]***. Since M. Friedman et al. ***[8]*** proposed a general model for solving a $n\times n$ fuzzy linear systems whose coefficients matrix is crisp and the right-hand side is a fuzzy number vector in 1998, many works have been done about how to deal with some advanced fuzzy linear systems such as dual fuzzy linear systems (*DFLS*), general fuzzy linear systems (*GFLS*), fully fuzzy linear systems (*FFLS*), dual fully fuzzy linear systems (*DFFLS*) and general dual fuzzy linear systems (*GDFLS*) see ***[9-12]***. However, for a fuzzy linear matrix equation which always has a wide use in control theory and control engineering, few works have been done in the past decades. In 2009, Allahviranloo et al. ***[13]*** discussed the fuzzy linear matrix equation (*FLME*) of the form $A\tilde{X}B=\tilde{C}$ where *A* and *B* are $m\times m$ and $n\times n$ crisp matrices and $\tilde{C}$ is an $m\times n$ arbitrary fuzzy numbers matrix. By using the parametric form of fuzzy number, they derived necessary and sufficient conditions for the existence of the set of fuzzy solutions and designed a numerical procedure for solving the solutions of original fuzzy matrix equation. In 2010, Gong et al. ***[14,15]*** investigated a class of fuzzy matrix equations $A\tilde{X}=\tilde{B}$ by means of the undetermined coefficients method, and studied least squares solutions of the inconsistent fuzzy matrix equation by using generalized inverses. In 2011, Guo et al. ***[16,17]*** studied the minimal fuzzy solution of linear fuzzy matrix equation $A\tilde{X}B=\tilde{C}$ and fuzzy Sylvester matrix equation $A\tilde{X}+\tilde{X}B=\tilde{C}$ based on triangular fuzzy numbers. Later, they ***[18, 19]*** investigated the fuzzy Sylvester matrix equation $A\tilde{X}+\tilde{X}B=\tilde{C}$ and the fully fuzzy matrix equations $\tilde{A}\tilde{X}\tilde{B}=\tilde{C}$ under LR fuzzy numbers. The symmetric fuzzy approximate solutions were firstly introduced by Allahviranloo et al. In 2011, Allahviranloo et al.***[20]*** obtained symmetric fuzzy approximate solutions of fuzzy linear systems by solving a crisp linear equation and a fuzzified interval equation which converted from the original fuzzy linear system $A\tilde{x}=\tilde{b}$. Subsequently, they ***[21]*** investigated the maximal and minimal symmetric solutions of fully fuzzy linear systems $\tilde{A}\tilde{x}=\tilde{b}$ by the same approach. These works give us a new viewpoint to understand the approximate solutions of fuzzy linear systems. In 2012, Guo et al. ***[22]*** considered the fuzzy symmetric solutions of fuzzy matrix equation $A\tilde{X}=\tilde{B}$. In this paper, we investigate the fuzzy symmetric solutions of linear fuzzy matrix equations $A\tilde{X}+\tilde{X}B=\tilde{C}$. The model is set up by this way, i.e., the authors first convert the fuzzy matrix equation to a fuzzy linear systems based on the Kronecker product of matrices, and then extend the fuzzy linear system into a a crisp system of linear equations and a fuzzified interval system of linear equations. Three types of fuzzy symmetric solutions of the fuzzy matrix equation are obtained by solving the model systems. Finally, some examples are given to illustrate the utility of our method. The structure of this paper is organized as follows. In Section 2, the author recalls the fuzzy number and their arithmetic operations. The model to the linear fuzzy matrix equation is proposed and the fuzzy symmetric solutions of the fuzzy matrix equation are obtained by solving the fuzzy linear system in Section 3 in detail. Some examples are given in Section 4 and the conclusion is drawn in Section 5.

II. PRELIMINARIES

Definition 1. ***[2]*** A fuzzy number u in parametric form is a pair ($\underline{u}, \overline{u}$) of functions ($\underline{u}(r), \overline{u}(r)$), $0\le r\le 1$ which satisfies the following requirements:

(1) $\underline{u}(r)$ is a bounded monotonic increasing left continuous function,

(2) $\overline{u}(r)$ is a bounded monotonic decreasing left continuous function,

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(3) $\underline{u}(r) \le \bar{u}(r), 0 \le r \le 1$.

Let E^1 be the set of all fuzzy numbers on R.

For an arbitrary real number $k \in R$ and the fuzzy numbers $x = (\underline{x}(r), \bar{x}(r)),\ y = (\underline{y}(r), y(r)) \in E^1,\ 0 \le r \le 1$,

(1) $x + y = (\underline{x}(r) + \underline{y}(r), \bar{x}(r) + \bar{y}(r))$,

(2) $x - y = (\underline{x}(r) - \bar{y}(r), \bar{x}(r) - \underline{y}(r))$,

(3) $kx = \begin{cases} (k\underline{x}(r), k\bar{x}(r)), & k \ge 0, \\ (k\bar{x}(r), k\underline{x}(r)), & k \le 0. \end{cases}$

Definition 2. The matrix system

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1m} \\ a_{21} & a_{22} & \cdots & a_{2m} \\ \cdots & \cdots & \cdots & \\ a_{m1} & a_{m2} & \cdots & a_{mm} \end{pmatrix} \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \cdots & \tilde{x}_{1n} \\ \tilde{x}_{21} & \tilde{x}_{22} & \cdots & \tilde{x}_{2n} \\ \cdots & \cdots & \cdots & \\ \tilde{x}_{m1} & \tilde{x}_{m2} & \cdots & \tilde{x}_{mn} \end{pmatrix} + \begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \cdots & \tilde{x}_{1n} \\ \tilde{x}_{21} & \tilde{x}_{22} & \cdots & \tilde{x}_{2n} \\ \cdots & \cdots & \cdots & \\ \tilde{x}_{m1} & \tilde{x}_{m2} & \cdots & \tilde{x}_{mn} \end{pmatrix} \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \cdots & \cdots & \cdots & \\ b_{n1} & b_{n2} & \cdots & b_{nn} \end{pmatrix} = \begin{pmatrix} \tilde{c}_{11} & \tilde{c}_{12} & \cdots & \tilde{c}_{1n} \\ \tilde{c}_{21} & \tilde{c}_{22} & \cdots & \tilde{c}_{2n} \\ \cdots & \cdots & \cdots & \\ \tilde{c}_{m1} & \tilde{c}_{m2} & \cdots & \tilde{c}_{mn} \end{pmatrix}, \quad (1)$$

where elements a_{ij}, b_{ij} are crisp numbers and $c_{ij} \in E^1$ are fuzzy numbers, called a fuzzy Sylvester matrix equations (*FSMEs*). Using matrix denoted, we have

$$A\tilde{X} + \tilde{X}B = \tilde{C}. \quad (2)$$

A fuzzy numbers matrix

$$\tilde{X} = (\underline{x}_{kj}(r), \bar{x}_{kj}(r)), \quad 1 \le k \le m, 1 \le j \le n, 0 \le r \le 1$$

is called a solution of the fuzzy linear matrix Equation (1) if $\tilde{X}$ satisfies (2).

III. METHOD FOR SOLVING

Definition 3. Suppose $\tilde{A} = (\underline{a}_{kj}(r), \bar{a}_{kj}(r)) \in E^{m\times n}$ and $\tilde{a}_j = (\tilde{a}_{1j}, \tilde{a}_{2j}, \cdots, \tilde{a}_{mj})^T$, $1 \le j \le n$, then the $m \times n$ dimensions fuzzy numbers vector

$$Vec(\tilde{A}) = \begin{pmatrix} \tilde{a}_1 \\ \tilde{a}_2 \\ \vdots \\ \tilde{a}_n \end{pmatrix} \quad (3)$$

is called the extension on column of the fuzzy matrix $\tilde{A}$.

At first, we convert the fuzzy linear matrix Equation (2) to a fuzzy system of linear equations based on the Kronecker product ***[23]*** of matrices.

Theorem 1 Let matrix A belong to $R^{m\times m}$, $\tilde{x} = (\underline{x}_{kj}(r), \bar{x}_{kj}(r))$ belong to $E^{m\times n}$, and matrix B belong to $R^{n\times n}$. Then

$$Vec(A\tilde{X}B) = (B^T \otimes A)Vec(\tilde{X}). \quad (4)$$

Proof. Denoting $\tilde{X} = (\tilde{x}_1, \tilde{x}_2, \ldots, \tilde{x}_n), \tilde{x}_j \in E^m$, matrix $B = (b_1, b_2, \cdots, b_n), b_j \in R^r$, we have

$$Vec(A\tilde{X}B) = Vec(A\tilde{X}b_1, A\tilde{X}b_2, \cdots, A\tilde{X}b_n) = \begin{pmatrix} A\tilde{X}b_1 \\ \vdots \\ A\tilde{X}b_n \end{pmatrix}.$$

Since

$$\begin{aligned} A\tilde{X}b_j &= (A\tilde{x}_1, A\tilde{x}_2, \ldots, A\tilde{x}_n)b_j \\ &= b_{1j}A\tilde{x}_1 + b_{2j}A\tilde{x}_2 + \ldots + b_{nj}A\tilde{x}_n \\ &= (b_{1j}A + b_{2j}A + \ldots + b_{nj}A)Vec(\tilde{X}), \end{aligned}$$

then

$$Vec(A\tilde{X}B) = (b_{ij}A)Vec(\tilde{X}) = (B^T \otimes A)Vec(\tilde{X}).$$

Theorem 2 Let matrix A belong to $R^{m\times m}$, $\tilde{x} = (\underline{x}_{kj}(r), \bar{x}_{kj}(r))$ belong to $E^{m\times n}$ and matrix B belong to $R^{n\times n}$. Then

$$Vec(A\tilde{X} + \tilde{X}B) = (I_n \otimes A + B^T \otimes I_m)Vec(\tilde{X}), \quad (5)$$

where I_n and I_m denote unit matrices with order *n* and order *m* respectively.

Proof. Setting $B = I_n$ in (3), we have

$$Vec(A\tilde{X}) = Vec(A\tilde{X}I_n) = (I_n \otimes A)Vec(\tilde{X}). \quad (6)$$

Similarly, the result

$$Vec(\tilde{X}B) = (B^T \otimes I_m)Vec(\tilde{X}) \quad (7)$$

is obvious when we replace A by I_m in (4).

We combine (6) and (7) and obtain the following conclusion

$$Vec(A\tilde{X} + \tilde{X}B) = (I_n \otimes A + B^T \otimes I_m)Vec(\tilde{X}).$$

Theorem 3 The fuzzy matrix $\tilde{X} \in E^{m\times n}$ is the solution of the fuzzy linear matrix equation (2) if and only if that $\tilde{x} = Vec(\tilde{X}) \in E^{mn}$ is the solution of the following linear fuzzy systems

$$G\tilde{x} = \tilde{y}, \quad (8)$$

where $G = (I_n \otimes A + B^T \otimes I_m)$, $\tilde{y} = Vec(\tilde{C})$.

Proof. Applying the extension operation to two sides of Eq. (2) and according to the Definition 3.1 and Theorem 3.2, the result is obvious.

For simplicity, we denote $p = mn$. Thus $G = (g_{ij})$ is a $p \times p$ real matrix and $\tilde{y}$ is a p fuzzy numbers vector.

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$$\begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1p} \\ g_{21} & g_{22} & \cdots & g_{2p} \\ \cdots & \cdots & \cdots & \\ g_{p1} & g_{p2} & \cdots & g_{pp} \end{pmatrix} \begin{pmatrix} \tilde{x}_1 \\ \tilde{x}_2 \\ \vdots \\ \tilde{x}_p \end{pmatrix} = \begin{pmatrix} \tilde{y}_1 \\ \tilde{y}_2 \\ \vdots \\ \tilde{y}_p \end{pmatrix}.$$

The following Definitions show what the fuzzy symmetric solutions of fuzzy Sylvester matrix equation are.

Definition 4. ***[24]*** The united solution set (*USS*), the tolerable solution set(*TSS*) and the controllable solution set(*CSS*) for the system (4) are respectively as follows:

$$X_{\exists\exists} = \{x \in R^p : Gx \cap y \neq \phi\},$$
$$X_{\forall\exists} = \{x \in R^p : Gx \subseteq y\},$$
$$X_{\exists\forall} = \{x \in R^p : Gx \supseteq y\}.$$

Definition 5. A fuzzy vector $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_p)^T$ given by $\tilde{x}_i = [\underline{x}_i(r), \overline{x}_i(r)], 1 \le i \le p, 0 \le r \le 1$ is called the minimal symmetric solution of the system (2) which is placed in *CSS* if for any arbitrary symmetric solution $\tilde{z} = (\tilde{z}_1, \tilde{z}_2, ..., \tilde{z}_p)^T$ which is placed in *CSS* that is $\tilde{x}(1) = \tilde{z}(1)$ we have

$$(\tilde{x} \subseteq \tilde{z}), \text{ i.e., } (\tilde{x}_i \subseteq \tilde{z}_i), \text{ i.e., } \sigma_{\tilde{x}_i} \subseteq \sigma_{\tilde{z}_i}, i = 1,2,..., p$$

where $\sigma_{\tilde{x}_i}$ and $\sigma_{\tilde{z}_i}$ are symmetric spreads of $\tilde{x}_i$ and $\tilde{z}_i$, respectively.

Definition 6. A fuzzy vector $\tilde{x} = (\tilde{x}_1, \tilde{x}_2, ..., \tilde{x}_p)^T$ given by $\tilde{x}_i = [\underline{x}_i(r), \overline{x}_i(r)], 1 \le i \le p, 0 \le r \le 1$ is called the maximal symmetric solution of the system (2) which is placed in *TSS* if for any arbitrary symmetric solution $\tilde{z} = (\tilde{z}_1, \tilde{z}_2, ..., \tilde{z}_p)^T$ which is placed in *TSS* that is $\tilde{x}(1) = \tilde{z}(1)$ we have

$$(\tilde{x} \supseteq \tilde{z}), \text{ i.e., } (\tilde{x}_i \supseteq \tilde{z}_i), \text{ i.e., } \sigma_{\tilde{x}_i} \supseteq \sigma_{\tilde{z}_i}, i = 1,2,..., p$$

where $\sigma_{\tilde{x}_i}$ and $\sigma_{\tilde{z}_i}$ are symmetric spreads of $\tilde{x}_i$ and $\tilde{z}_i$, respectively.

Secondly, in order to solve the fuzzy linear matrix Equation (2), we need to consider the fuzzy linear systems (8). Applied Allahviranloo`s method in ***[9]***, we extend (8) into to a crisp system of linear equations and a fuzzified interval system of linear equations to obtain its fuzzy symmetric solutions.

Theorem 4. ***[22]*** The fuzzy linear system (8) can be extended into an $p \times p$ crisp function system of linear equations

$$\begin{pmatrix} g_{11} & g_{12} & \cdots & g_{1p} \\ g_{21} & g_{22} & \cdots & g_{2p} \\ \cdots & \cdots & \cdots & \\ g_{p1} & g_{p2} & \cdots & g_{pp} \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{pmatrix} = \begin{pmatrix} \tilde{y}_1(1) \\ \tilde{y}_2(1) \\ \vdots \\ \tilde{y}_p(1) \end{pmatrix} \qquad (9)$$

and

$$\begin{cases} g_{11}(x_1 - \alpha_1(r), x_1 + \alpha_1(r)) + \cdots + \\ g_{1p}(x_p - \alpha_1(r), x_p + \alpha_1(r)) = (\underline{y}_1(r), \overline{y}_1(r)), \\ g_{21}(x_1 - \alpha_2(r), x_1 + \alpha_2(r)) + \cdots + \\ g_{2p}(x_p - \alpha_2(r), x_p + \alpha_2(r)) = (\underline{y}_2(r), \overline{y}_2(r)), \\ \vdots \\ g_{p1}(x_1 - \alpha_p(r), x_1 + \alpha_p(r)) + \cdots + \\ g_{pp}(x_p - \alpha_p(r), x_p + \alpha_p(r) = (\underline{y}_p(r), \overline{y}_p(r)), \end{cases} \qquad (10)$$

where $\tilde{y}_i(1) \in R, i = 1,2,..., p$ and $\alpha_i(r), i = 1,2,..., p$ are unknown spreads.

Now, we solve the crisp linear system (9) to obtain $x_i, i = 1,2,..., p$ that is existed uniquely since $\det(G) = \det(B^T \otimes A) \neq 0$ and solve the interval equations (7) to obtain $\alpha_i(r), i = 1,2,..., p$.

So, without loss of generality and for simplicity to express the theory, we assumed that the coefficients matrix G is positive. Then, *i*th equation of interval system (10) is

$$\begin{aligned} & g_{i1}(x_1 - \alpha_i(r), x_1 + \alpha_i(r)) + \cdots + \\ & g_{ip}(x_p - \alpha_i(r), x_p + \alpha_i(r)) = (\underline{y}_i(r), \overline{y}_i(r)), \end{aligned} \qquad (11)$$

it can be rewritten in parametric form:

$$\sum_{j=1}^{p} gi_j(x_j - \alpha_i(r)) = \underline{y}_i(r), i = 1,2,..., p \qquad (12)$$

and

$$\sum_{j=1}^{p} gi_j(x_j + \alpha_i(r)) = \underline{y}_i(r), i = 1,2,..., p. \qquad (13)$$

So, after some computations, we replace $\alpha_i(r)$ with $\alpha_{i1}(r), i = 1,2,..., p$ in (12) and replace $\alpha_i(r)$ with $\alpha_{i2}(r), i = 1,2,..., p$ in (13), then the Eqs.(12) and (13) are transformed respectively to

$$\alpha_{i1}(r) = f_1(x_1, \cdots, x_p, g_{i1}, \cdots g_{ip}, \underline{y}_i(r)), i = 1,2,..., p$$

and

$$\alpha_{i2}(r) = f_2(x_1, \cdots, x_p, g_{i1}, \cdots g_{ip}, \overline{y}_i(r)), i = 1,2,..., p.$$

However, $\alpha_{i1}(r)$ is a function of $x_1, \cdots, x_p, g_{i1}, \cdots g_{ip}, \underline{y}_i(r), i = 1,2,..., p$ and $\alpha_{i2}(r)$ is a function of $x_1, \cdots, x_p, g_{i1}, \cdots g_{ip}, \overline{y}_i(r), i = 1,2,..., p$ such that $\alpha_{i1}(r)$ and $\alpha_{i2}(r)$ are obtained spreads of *i*th equation in system (10). Perhaps, $\alpha_{i1}(r)$ and $\alpha_{i2}(r)$ do not satisfy the rest of interval Equations (10). Therefore, we should determine the reasonable spreads according to decision makers. To this end, three type of spreads are proposed as follows:

$$\alpha_L(r) = \min\{\alpha_{i1}(r), \alpha_{i2}(r)\}, i = 1,2, \cdots, p, 0 \le r \le 1, \qquad (14)$$

$$\alpha_U(r) = \max\{\alpha_{i1}(r), \alpha_{i2}(r)\}, i = 1,2, \cdots, p, 0 \le r \le 1, \qquad (15)$$

$$\begin{aligned} & \alpha_\lambda(r) = \lambda \alpha_U(r) + (1 - \lambda)\alpha_L(r), \\ & i = 1,2, \cdots, p, 0 \le r \le 1, \lambda \in [0,1] \end{aligned}. \qquad (16)$$

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Hence, by such computations, the fuzzy vector solution of system (2) under proposed spreads (11)-(13) will be as follows. For $i=1,2,\cdots,p, 0\le r\le 1, \lambda\in[0,1]$

$$X_L=(\tilde{x}_1(r),\cdots,\tilde{x}_p(r))^T, \tilde{x}_i(r)=(x_i-\alpha_L(r), x_i+\alpha_L(r)), \quad (17)$$

$$X_U=(\tilde{x}_1(r),\cdots,\tilde{x}_p(r))^T, \tilde{x}_i(r)=(x_i-\alpha_U(r), x_i+\alpha_U(r)), \quad (18)$$

$$X_\lambda=(\tilde{x}_1(r),\cdots,\tilde{x}_p(r))^T, \tilde{x}_i(r)=(x_i-\alpha_\lambda(r), x_i+\alpha_\lambda(r)). \quad (19)$$

Now, we show that our method always gives us a fuzzy vector solution provided that the right-hand side of system (2) be a triangular fuzzy vector with non-zero left and right spreads.

From formulas (14)-(19), the following result is apparent.

Theorem 5. Let the right hand side of the system (2) be

$$\tilde{y}(r)=(\tilde{y}_1(r),\cdots,\tilde{y}_p(r))^T,$$

where $\tilde{y}_i(r)=[\underline{y}_i(1)-\sigma_i(1-r), \bar{y}_i(r)+\beta_i(1-r)], i=1,2,...,p$ and let $\alpha_L(r)$, $\alpha_U(r)$ and $\alpha_\lambda(r)$ be defined by (14)-(16), then $\alpha_L(r)$, $\alpha_U(r)$ and $\alpha_\lambda(r)$ are positive for all $0\le r\le 1, \lambda\in[0,1]$ such that

$$\alpha_L(r)=\min\{\frac{\sigma_i(1-r)}{\sum|g_{ij}|}, \frac{\beta_i(1-r)}{\sum|g_{ij}|}\}, \quad (20)$$

$$\alpha_U(r)=\max\{\frac{\sigma_i(1-r)}{\sum|g_{ij}|}, \frac{\beta_i(1-r)}{\sum|g_{ij}|}\}, \quad (21)$$

$$\alpha_\lambda(r)=\lambda\max\{\frac{\sigma_i(1-r)}{\sum|g_{ij}|}, \frac{\beta_i(1-r)}{\sum|g_{ij}|}\}+ (1-\lambda)\min\{\frac{\sigma_i(1-r)}{\sum|g_{ij}|}, \frac{\beta_i(1-r)}{\sum|g_{ij}|}\}. \quad (22)$$

Theorem 6. Let us consider spreads (20)-(22) and corresponding solutions $\tilde{X}_L$ and $\tilde{X}_U$, then we get

(1) $\tilde{X}_L$ is maximal symmetric solution in *TSS*,

(2) $\tilde{X}_U$ is minimal symmetric solution in *CSS*.

Algorithm.

(1) We convert the fuzzy linear matrix Equation (2) to a fuzzy system of linear Equations (6) according to the Kronecker product of matrices.

(2) We solve system (9) to obtain its crisp solution, i.e. , $x_i, i=1,2,...,p$.

(3) By applying crisp solution (solution of 1-cut), system (5) is transformed into the system of interval equations (7).

(4) The spread of all elements of fuzzy vector solution will be obtained by solving system (10), whereas, spreads are named as $\alpha_{i1}(r), \alpha_{i2}(r), i=1,2,...,p, 0\le r\le 1,$ respectively.

(5) The symmetric spreads can be assessed by using (14)-(16).

(6) The fuzzy vector solutions are derived by (17)-(19).

IV. NUMERICAL EXAMPLES

In this section, we work out an numerical example to illustrate our proposed method.

Example Consider the fuzzy linear matrix systems:

$$\begin{pmatrix}1 & 0\\ -1 & 1\end{pmatrix}\begin{pmatrix}\tilde{x}_{11} & \tilde{x}_{12} & \tilde{x}_{13}\\ \tilde{x}_{21} & \tilde{x}_{22} & \tilde{x}_{23}\end{pmatrix}+\begin{pmatrix}\tilde{x}_{11} & \tilde{x}_{12} & \tilde{x}_{13}\\ \tilde{x}_{21} & \tilde{x}_{22} & \tilde{x}_{23}\end{pmatrix}\begin{pmatrix}1 & 0 & -1\\ 0 & -1 & 1\\ 1 & 2 & 0\end{pmatrix}$$
$$=\begin{pmatrix}(1+r,3-r) & (5+r,7-r) & (2+r,4-r)\\ (6+r,8-r) & (3+r,\ 5-r) & (r,\ 2-r)\end{pmatrix}.$$

By Theorems 1. and 2., the original fuzzy matrix equation is equivalent to the following fuzzy linear system $G\tilde{x}=\tilde{y}$, i.e. ,

$$\begin{pmatrix}2 & 0 & 0 & 0 & 1 & 0\\ -1 & 1 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 2 & 0\\ 0 & 0 & -1 & -1 & 0 & 2\\ -1 & 0 & 1 & 0 & 1 & 0\\ 0 & -1 & 0 & 1 & -1 & 1\end{pmatrix}\begin{pmatrix}\tilde{x}_{11}\\ \tilde{x}_{21}\\ \tilde{x}_{12}\\ \tilde{x}_{22}\\ \tilde{x}_{13}\\ \tilde{x}_{23}\end{pmatrix}=\begin{pmatrix}(1+r,3-r)\\ (6+r,8-r)\\ (5+r,7-r)\\ (3+r,5-r)\\ (2+r,4-r)\\ (r,2-r)\end{pmatrix}.$$

Using the approach in [8] of Friedman, the above fuzzy linear system can be extended into the following function system of linear equations $SX(r)=Y(r)$. After some computations, we know that the exact solution of the original fuzzy linear matrix equation $A\tilde{X}+\tilde{X}B=\tilde{C}$ is

$$\tilde{X}=\begin{pmatrix}\tilde{x}_{11} & \tilde{x}_{12} & \tilde{x}_{13}\\ \tilde{x}_{21} & \tilde{x}_{22} & \tilde{x}_{23}\end{pmatrix}$$
$$=\frac{1}{4}\begin{pmatrix}(-3+r,-1-r) & (-3+r,-1-r) & (10+2r,\ 14-2r)\\ (11+r,\ \ 13-r) & (15-r,\ 13+r) & (12+2r,16-2r)\end{pmatrix},$$

which admits a weak fuzzy solution since $\tilde{x}_{22}$ is not fuzzy numbers.

Form Theorem 3., 1-cut of extended system is

$$\begin{pmatrix}2 & 0 & 0 & 0 & 1 & 0\\ -1 & 1 & 0 & 0 & 0 & 1\\ 0 & 0 & 0 & 0 & 2 & 0\\ 0 & 0 & -1 & -1 & 0 & 2\\ -1 & 0 & 1 & 0 & 1 & 0\\ 0 & -1 & 0 & 1 & -1 & 1\end{pmatrix}\begin{pmatrix}x_{11}\\ x_{21}\\ x_{12}\\ x_{22}\\ x_{13}\\ x_{23}\end{pmatrix}=\begin{pmatrix}2\\ 4\\ 6\\ 7\\ 3\\ 1\end{pmatrix}.$$

Therefore, the crisp solution is $(-1/2, 11/4, 1/2, 3, 3, 15/4)^T$.

Now the system of interval Equation (10) is as follows:

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$$\begin{cases} 2(-1/2-\alpha_1(r),-1/2+\alpha_1(r))+(3-\alpha_1(r),3+\alpha_1(r)) \\ =(1+r,3-r), \\ (1/2-\alpha_2(r),1/2+\alpha_2(r))+(11/4-\alpha_2(r),11/4+\alpha_2(r)) \\ +(15/4-\alpha_2(r),15/4+\alpha_2(r))=(6+r,8-r), \\ 2(3-\alpha_3(r),3+\alpha_3(r))=(5+r,7-r), \\ -(1/2-\alpha_4(r),1/2+\alpha_4(r))-(3-\alpha_4(r),3+\alpha_4(r)) \\ -2(15/4-\alpha_4(r),15/4+\alpha_4(r))=(3+r,5-r), \\ -(-1/2-\alpha_5(r),-1/2+\alpha_5(r))+(1/2-\alpha_5(r),1/2+\alpha_5(r)) \\ +(3-\alpha_5(r),3+\alpha_5(r))=(2+r,4-r), \\ -(11/4-\alpha_6(r),-11/4+\alpha_6(r))+(15/4-\alpha_6(r),11/4+\alpha_6(r)) \\ =(r,2-r). \end{cases}$$

Hence, the following results are obtained for all $r\in[0,1]$ as

$$\alpha_{11}(\mathrm{r})=\alpha_{12}(\mathrm{r})=\alpha_{15}(\mathrm{r})=\frac{1\text{-r}}{3},\ \alpha_{13}(\mathrm{r})=\frac{1\text{-r}}{2},\ \alpha_{14}(\mathrm{r})=\alpha_{16}(\mathrm{r})=\frac{1\text{-r}}{4},$$

$$\alpha_{21}(\mathrm{r})=\alpha_{22}(\mathrm{r})=\alpha_{25}(\mathrm{r})=\frac{1\text{-r}}{2},\ \alpha_{23}(\mathrm{r})=\frac{1\text{-r}}{3},\ \alpha_{24}(\mathrm{r})=\alpha_{26}(\mathrm{r})=\frac{1\text{-r}}{4},$$

and applying (14)-(16) we get for all $0\le r\le 1,\lambda\in[0,1]$

$$\alpha_L(r)=\frac{1-r}{4},\ \alpha_U(r)=\frac{1-r}{2},\ \alpha_\lambda(r)=\frac{(1-r)(1-\lambda)}{4}.$$

Thus the fuzzy symmetric solutions of the Eq.(8) are obtained as follows:

$$\tilde{X}_L=\begin{pmatrix} (-1/2-\frac{1-r}{4},-1/2+\frac{1-r}{4}) \\ (\frac{11}{4}-\frac{1-r}{4},\frac{11}{4}+\frac{1-r}{4}) \\ (1/2-\frac{1-r}{4},1/2+\frac{1-r}{4}) \\ (3-\frac{1-r}{4},3+\frac{1-r}{4}) \\ (3-\frac{1-r}{4},3+\frac{1-r}{4}) \\ (\frac{15}{4}-\frac{1-r}{4},\frac{15}{4}+\frac{1-r}{4}) \end{pmatrix},\quad \tilde{X}_U=\begin{pmatrix} (-1/2-\frac{1-r}{2},-1/2+\frac{1-r}{2}) \\ (\frac{11}{4}-\frac{1-r}{2},\frac{11}{4}+\frac{1-r}{2}) \\ (1/2-\frac{1-r}{2},1/2+\frac{1-r}{2}) \\ (3-\frac{1-r}{2},3+\frac{1-r}{2}) \\ (3-\frac{1-r}{2},3+\frac{1-r}{2}) \\ (\frac{15}{4}-\frac{1-r}{2},\frac{15}{4}+\frac{1-r}{2}) \end{pmatrix},$$

$$\tilde{X}_\lambda=\begin{pmatrix} (-1/2-\frac{(1-r)(1-\lambda)}{4},-1/2+\frac{(1-r)(1-\lambda)}{4}) \\ (\frac{11}{4}-\frac{(1-r)(1-\lambda)}{4},\frac{11}{4}+\frac{(1-r)(1-\lambda)}{4}) \\ (1/2-\frac{(1-r)(1-\lambda)}{4},1/2+\frac{(1-r)(1-\lambda)}{4}) \\ (3-\frac{(1-r)(1-\lambda)}{4},3+\frac{(1-r)(1-\lambda)}{4}) \\ (3-\frac{(1-r)(1-\lambda)}{4},3+\frac{(1-r)(1-\lambda)}{4}) \\ (\frac{15}{4}-\frac{(1-r)(1-\lambda)}{4},\frac{15}{4}+\frac{(1-r)(1-\lambda)}{4}) \end{pmatrix}.$$

According to Theorem 2., we know that the fuzzy approximate symmetric solutions of the original fuzzy matrix equation are

$$\tilde{X}_L=\begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \tilde{x}_{13} \\ \tilde{x}_{21} & \tilde{x}_{22} & \tilde{x}_{23} \end{pmatrix}$$

$$=\begin{pmatrix} (-\frac{1}{2}-\frac{1-r}{4},-\frac{1}{2}+\frac{1-r}{4}) & (\frac{1}{2}-\frac{1-r}{4},\frac{1}{2}+\frac{1-r}{4}) \\ (3-\frac{1-r}{4},3+\frac{1-r}{4}) & \\ (\frac{11}{4}-\frac{1-r}{4},\frac{11}{4}+\frac{1-r}{4}) & (3-\frac{1-r}{4},3+\frac{1-r}{4}) \\ (\frac{15}{4}-\frac{1-r}{4},\frac{15}{4}+\frac{1-r}{4}) & \end{pmatrix},$$

$$\tilde{X}_U=\begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \tilde{x}_{13} \\ \tilde{x}_{21} & \tilde{x}_{22} & \tilde{x}_{23} \end{pmatrix}$$

$$=\begin{pmatrix} (-\frac{1}{2}-\frac{1-r}{2},-\frac{1}{2}+\frac{1-r}{2}) & (\frac{1}{2}-\frac{1-r}{2},\frac{1}{2}+\frac{1-r}{2}) \\ (3-\frac{1-r}{2},3+\frac{1-r}{2}) & \\ (\frac{11}{4}-\frac{1-r}{2},\frac{11}{4}+\frac{1-r}{2}) & (3-\frac{1-r}{2},3+\frac{1-r}{2}) \\ (\frac{15}{4}-\frac{1-r}{2},\frac{15}{4}+\frac{1-r}{2}) & \end{pmatrix},$$

and

$$\tilde{X}_\lambda=\begin{pmatrix} \tilde{x}_{11} & \tilde{x}_{12} & \tilde{x}_{13} \\ \tilde{x}_{21} & \tilde{x}_{22} & \tilde{x}_{23} \end{pmatrix}=$$

$$\begin{pmatrix} (-\frac{1}{2}-\frac{(1-r)(1-\lambda)}{4},-\frac{1}{2}+\frac{(1-r)(1-\lambda)}{4}) & (\frac{1}{2}-\frac{(1-r)(1-\lambda)}{4},\frac{1}{2}+\frac{(1-r)(1-\lambda)}{4}) \\ (3-\frac{(1-r)(1-\lambda)}{4},3+\frac{(1-r)(1-\lambda)}{4}) & \\ (\frac{11}{4}-\frac{(1-r)(1-\lambda)}{4},\frac{11}{4}+\frac{(1-r)(1-\lambda)}{4}) & (3-\frac{(1-r)(1-\lambda)}{4},3+\frac{(1-r)(1-\lambda)}{4}) \\ (\frac{15}{4}-\frac{(1-r)(1-\lambda)}{4},\frac{15}{5}+\frac{(1-r)(1-\lambda)}{4}) & \end{pmatrix},$$

respectively.

V. CONCLUSION

In this work we investigated the approximate solutions of linear fuzzy matrix equations $A\tilde{X}+\tilde{X}B=\tilde{C}$ in which A and B are $m\times m$ and $n\times n$ crisp matrices and $\tilde{C}$ is an $m\times n$ arbitrary fuzzy numbers matrix with nonzero spreads. The model was proposed in this way, that is, we converted the fuzzy matrix equation to a fuzzy system of linear equations based on the Kronecker product at first, and then extended the fuzzy linear system into a crisp system of linear equations and a fuzzified interval system of linear equations. The fuzzy symmetric solutions of the linear fuzzy matrix equation were derived from solving two crisp systems of linear equations. Numerical example showed that our method is effective to solve this type of fuzzy matrix equations.

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