Hybrid Projective Synchronization between Non-identical Fractional Order Chaotic Systems

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Abstract—This paper deals with hybrid projective synchronization between a fractional order Volta chaotic system and Genesio Tesi chaotic system by using tracking control and stability theory of fractional order system. An effective controller is designed to synchronize these two systems. Numerical simulations have been done by using MATLAB. Numerical solutions via Grunwald-Letnikov method has been used in matlab. Numerical results show that method is effective and feasible.

Index Terms-Fractional order derivatives, Hybrid Projective Synchronization, Tracking control method.

I. INTRODUCTION

Synchronization is the most popular phenomena which can be understood within the unifying framework of the nonlinear sciences. It has its own importance in the field of non-linear dynamics. Synchronization in the language of the nonlinear dynamics is defined as an adjustment of rhythms of oscillating objects due to their weak interaction [38]. Many different objects presenting Nature that periodically repeat its state, i.e., oscillate, and that these systems are generally not isolated but interacting with each other, it is easy to recognize that synchronization must be acting on a huge number of structures at different scales. From an audience in which people tend to accommodate their claps to sound in unison, to thousands of pacemaker cells in our heart tissue spiking their action potentials together to induce a beat in our heart every few seconds, the objects able to experience sync cover a wide spectrum. In1673, It was first described by Dutch scientist Huygens, who observed weak synchronization of two double pendulum clocks [22]. It is mainly studied in regular (periodic) and complex (chaotic or quasiperiodic) dynamical systems. A great boost was that to shake the investigation in synchronization came in the early nineties and made it one of the hottest research topics in nonlinear science as it is considered today when in February of1990, Lou Pecora and Tom Carroll introduced a paper entitled Synchronization in Chaotic Systems [34]. They demonstrated that chaotic synchronization could be achieved by driving or replacing one of the variables of a chaotic system with a variable of another similar chaotic device. After this discovery, mathematicians started showing interest in synchronization of chaotic systems. In the last two decades considerable research has been done in non-linear dynamical systems and their various properties. One of the most important properties is synchronization. This problem has received the great attention in the literature due to its importance in engineering and physical sciences, as well as in the challenging biological and social entities [38,16,42]. Synchronization techniques have been improved in recent years and many different methods are applied theoretically as well as experimentally to synchronize the chaotic-systems including adaptive control [26,12], back stepping design [44, 45], active control [6, 21], nonlinear control [7,33] and observer based control method [13,46]. Using these methods, numerous synchronization problem of well-known chaotic systems such as Lorenz, Chen, L’u and R’ossler system have been worked on by many researchers. In sequel to the study of chaotic systems, chaotic dynamics of fractional order systems have also been studied popularly. Since many real objects are generally fractional, so fractional calculus opens wide ways to describe a real object more accurately than the classical integer methods. So the fractional order methods become global and allow greater degree of flexibility in the study dynamical models. Due to advantage over integer methods it has a lot of important applications in the various fields like control theory [41, 35], viscoelastic [3], diffusion [24, 9], bioengineering [29], dielectric polarization [43], electrode-electrolyte polarization[23], electromagnetic waves [20], medicine [17] etc. Chaotic dynamics of fractional order systems is becoming an important field of investigation in nonlinear dynamics. Although the fractional calculus is more than three century old subject, yet in past few years it has increased rapidly. Analysis of fractional order dynamical systems has been studied by authors in [31,32,39]. Geometric and physical interpretation of fractional integration and fractional differentiation has also been studied by Podlubny [40]. In the continuation of study of chaos in fractional order dynamical systems, one of the important property synchronization of dynamical systems of fractional order has also got much attention. We can see many work on chaos in fractional order system in Chen’s system [25], Volta system [37], R’ossler system [8], Chua system [19], Duffing oscillators [15], cellular network [1], Lorenz system [18] etc. Various kind of synchronization between identical as well as non-identical fractional order systems has been presented in [50,49,28,10,4]. Also, several types of chaos synchronization are well known, which include complete synchronization (CS), ant synchronization (AS), phase synchronization, generalized synchronization (GS), projective synchronization (PS), and modified projective synchronization (MPS). Among all type of synchronization, Projective synchronization (PS) has been extensively considered because it can ob-
tain faster communication. The drive and response system could be synchronized up to a scaling factor in projective synchronization. In this continuation of study, in order to increase the degree of secrecy for secure communications, the same scaling factor in PS can be replaced by vector function factor. Moreover, in Hybrid Projective Synchronization, the same scaling factor is replaced by a vector function factor. HPS is actually generalization of PS, which is more useful in secure communication as compare to PS. There are many works done in PS for fractional order chaotic systems [47,46,27,5]. In this paper we have studied HPS between two different fractional order chaotic systems. Here we have used tracking control method to achieve hybrid projective synchronization between Volta and Genesio Tesi chaotic systems of fractional order. Numerical simulations have been done by using MATLAB. For fractional order system we have used Gr¨unwald-Letnikov method [39].

II. A GLIMPSE OF FRACTIONAL CALCULUS

In this section we mention some fundamental properties and definitions of fractional order derivatives. Fractional calculus is a generalization of integration and differentiation to non-integer order fundamental operator \( aD_t^\alpha \), where \( a \) and \( t \) are the limits of the operation and \( \alpha \) is the fractional order which can be a complex number, \( R(\alpha) \) denotes the real part of \( \alpha \). The continuous-integro-differential operator is defined as

\[
aD_t^\alpha = \begin{cases} 
\frac{d^\alpha}{dt^\alpha}, & R(\alpha) > 0, \\
1 & R(\alpha) = 0, \\
\int_a^t d\tau^{-\alpha} & R(\alpha) < 0.
\end{cases}
\]

The three definitions used for general fractional differentiability Gr¨unwald-Letnikov (GL) definition, theRiemann-Liouville (RL) and Caputo’s definition [39]. The GL definition is given as:

\[
aD_t^\alpha f(t) = \lim_{h \to 0} h^{-\alpha} \left[ \frac{t-a}{h} \sum_{j=0}^{[\frac{t-a}{h}]} (-1)^j \binom{\alpha}{j} f(t-jh) \right]
\]

where \([ \cdot ]\) denotes the integer part. And the RL definition is given by

\[
aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau,
\]

\( n-1 < \alpha < n, \)

Where \( \Gamma(.) \) is the gamma function. The Caputo fractional Derivative is

\[
aD_t^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{\alpha-n+1}} d\tau.
\]

For numerical calculations of fractional-order derivatives, we have used Gr¨unwald-Letnikov method which’s derived from Gr¨unwald-Letnikov definition. Which is also so-called Power Series Expansion method [11, 36].

A. Stability of Fractional order systems

An autonomous system is asymptotically stable if

\[
|\arg \lambda| > \frac{q\pi}{2}
\]

is satisfied for all eigenvalues \( \lambda \) of matrix A. Also this system is stable iff

\[
|\arg \lambda| \geq \frac{q\pi}{2}
\]

is satisfied for all eigenvalues of a matrix A and those critical eigenvalues which satisfy

\[
|\arg \lambda| > \frac{q\pi}{2}
\]

have geometric multiplicity one [30].

III. METHODOLOGY AND PROBLEM FORMULATION FOR HPS BETWEEN FRACTIONAL ORDER SYSTEMS

In this section we put a glimpse of methodology and problem formulation for hybrid projective synchronization between fractional order chaotic systems via tracking control. Consider the following \( n \)-dimensional fractional order chaotic system as drive (master) system

\[
d\frac{q}{dt} x = f(x)
\]

where \( x \in \mathbb{R}^n \), fractional order \( q= (q_1, q_2, ..., q_n) \): (0 < \( q_i < 1 \)) may be unequal. \( f : \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a differentiable function. Now, consider the following \( n \)-dimensional chaotic system of fractional order as:

\[
d\frac{q}{dt} y = g(y)
\]

\( n-1 < q < n \).
Where \( y \in \mathbb{R}^n \) and \( g: \mathbb{R}^n \rightarrow \mathbb{R}^n \) is a differentiable function and construct the following response system:

\[
\frac{d^q y}{dt^q} = g(y) + \Psi(y, x)
\]

Where \( \Psi(y, x) \) is vector controller to be designed via tracking control method.

**Definition 1**: Hybrid Projective Synchronization (HPS) between two chaotic system achieved if there exist an \( n \times n \) matrix \( A \) such that

\[
\lim_{t \to \infty} \|e(t)\| = \lim_{t \to \infty} \|Ay - x\| = 0
\]

Where \( \| \cdot \| \) is the Euclidean norm.

In order to achieve the hybrid projective synchronization between two fractional order chaotic systems, we choose the fractional order chaotic system (1) as a drive system and construct a response system as follows:

\[
\frac{d^q y}{dt^q} = A^{-1} [f(Ay) + \Psi(y, x)]
\]

(4)

Where \( A^{-1} \) is the inverse matrix of the invertible matrix \( A \) and \( y \in \mathbb{R}^n \) are state vector of the response system (2) and \( \Psi(y, x) \) is controller which will be designed. Now define the HPS errors between two given systems (1) and (4) as

\[
e(t) = Ay - x;
\]

where \( e = (e_1, e_2, ..., e_n)^T \) and \( A = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \).

So,

\[
e_i = (\sum_{j=1}^{n} a_{ij} y_j) - x_i, \quad \text{where}
\]

\((i, j = 1, 2, 3...n)\)

Let

\( f(Ay) - f(x) = F(x, e) \)

Now, we assume that the error vectors \( e \) can be divided into \( e_c = (e_{c1}, e_{c2}, ..., e_{cn})^T \) and \( e_n = (e_{n1}, e_{n2}, ..., e_{nn})^T \) such that \( F(x; e) \) has the following form

\[
F(x, e) = \begin{pmatrix} B_n e_n + h_1(x, e_n, e_m) \\ B_m e_m + h_2(x, e_n, e_m) \end{pmatrix}
\]

where \( h_1(x, e_n, e_m) \in \mathbb{R}^k \),

\( h_2(x, e_n, e_m) \in \mathbb{R}^{l-k} \),

\( h_2(x, e_n, e_m) \in \mathbb{R}^{l-k} \) and

\[
\lim_{t \to \infty} h_1(x, e_n, e_m) = 0
\]

\[
B_n \in \mathbb{R}^{k \times k}, \quad B_m \in \mathbb{R}^{l-k \times l-k}
\]

are respectively and real constant matrix.

Now, following theorem is based on the stability of fractional order chaotic systems, which gives the final destination of the problem formulation.

**Theorem 1**: If controller \( \Psi(y, x) \) in response system is

\[
\Psi(y, x) = \begin{pmatrix} \Psi_n(y, x) \\ \Psi_m(y, x) \end{pmatrix} = \begin{pmatrix} \Lambda_n e_n - h_1(x, e_n, e_m) \\ \Lambda_m e_m - h_2(x, e_n, e_m) \end{pmatrix},
\]

\( \Lambda_n \in \mathbb{R}^{k \times k}, \quad \Lambda_m \in \mathbb{R}^{l-k \times l-k} \) are suitable chosen constant matrices. If all eigenvalues of \( B_n + \Lambda_n \) and \( B_m + \Lambda_m \) satisfy

\[
| \arg \lambda | > \frac{q \pi}{2}.
\]

Then HPS between drive and response system can be achieved.

**Proof**: With the help of stability theory [30] and definition of hybrid projective synchronization, we can proof this theorem easily.

**IV. SYSTEM DESCRIPTION**

**A. Volta System of Fractional order**

The Volta system is a set of three coupled ordinary differential equations exhibiting chaotic behaviour for certain values of parameters [37]. The equations of system are:

\[
\begin{align*}
\frac{d^q x_1}{dt^q} &= -x_1 - ax_2 - x_3 x_2, \\
\frac{d^q x_2}{dt^q} &= -x_2 - bx_1 - x_3 x_1, \\
\frac{d^q x_3}{dt^q} &= cx_3 + x_3 x_2 + 1.
\end{align*}
\]

Where \( a, b, c \) and \( c \) are constant parameters that control the behavior of the system. For system parameters \( (a, b, c) = (19, 11, 0.73) \), chaos can be observed in Volta System of fractional order \( (q_1 = q_2 = q_3) \).

Figures (see fig.2, fig.3, fig.4, fig.5, fig.6 and fig.7) show the chaotic behaviour of fractional order Volta system with different values of orders.
B. Genesio Tesi System of Fractional Order

The Genesio Tesi chaotic system consist of three fractional order differential equations with orders \( q_1 \), \( q_2 \), and \( q_3 \) respectively [14]

\[
\begin{align*}
\frac{d^{q_1}}{dt^{q_1}} y_1 &= y_2, \\
\frac{d^{q_2}}{dt^{q_2}} y_2 &= y_3, \\
\frac{d^{q_3}}{dt^{q_3}} y_3 &= -b_1 y_1 - b_2 y_2 - b_3 y_3 - b_4 y_1^2.
\end{align*}
\]
Where $b_1 = 1.1$, $b_2 = 1.1$, $b_3 = 0.45$, and $b_4 = 1$. We can vary values of $q_1$, $q_2$, and $q_3$ accordingly. Figures (see fig.8, fig.9, fig.10, fig.11, fig.12 and fig.13) show chaotic behavior of the system with different values of $q_1$, $q_2$, and $q_3$ respectively.

V. ILLUSTRATION

If Volta system drives the Genesio Tesi system, then according to methodology, response system is

$$\frac{d^q y}{dt^q} = A^{-1}[f(Ay) + \Psi(y, x)]$$

which leads to response system as follows.
\[
\begin{aligned}
&\frac{d^q y_1}{dt^q} = F(x,e) + \Psi(y,x) \\
&\frac{d^q y_2}{dt^q} = A^{-1} [f(Ay) + \Psi(y,x)] \\
&\frac{d^q y_3}{dt^q} = \left\{\begin{array}{l}
-\sum_{j=1}^{3} a_{2j}y_{j} - \sum_{j=1}^{3} a_{3j}y_{j} \\
-\sum_{j=1}^{3} a_{2j}y_{j} - \sum_{j=1}^{3} a_{3j}y_{j} \\
+c\sum_{j=1}^{3} a_{j}y_{j} - \sum_{j=1}^{3} a_{3j}y_{j}
\end{array}\right\} + \Psi(y,x)
\end{aligned}
\]  

Yields
\[
\begin{pmatrix}
\frac{d^q y_1}{dt^q} \\
\frac{d^q y_2}{dt^q} \\
\frac{d^q y_3}{dt^q}
\end{pmatrix} = A^{-1} [f(Ay) + \Psi(y,x)]
\]

which yields,
\[
F(x,e) = \begin{pmatrix}
-e_1 a e_2 - e_2 e_3 - e_2 x_3 - e_3 x_1 \\
-e_2 b e_1 - e_1 e_3 - e_1 x_3 - e_3 x_1 \\
c e_3 + e_1 x_2 - e_1 e_2 + x_1 e_2
\end{pmatrix}
\]

(10)

Since we have taken three dimensional systems, so we Choose following
\[
\bar{e}_1 = e_1, \quad \bar{e}_2 = (e_2, e_3)^T, \quad B_1 = (-1)e_1
\]

\[
B_2 = \begin{pmatrix}
-1 & 0 \\
0 & c
\end{pmatrix}
\]

\[
h_1(x, \bar{e}_1, \bar{e}_2) = (-ae_2 - e_2 e_3 - e_2 x_3 - e_2 x_2),
\]

\[
h_{21}(x, \bar{e}_1, \bar{e}_2) = \begin{pmatrix}
-e_1 e_3 - e_1 x_3 \\
e_2 e_1 + e_1 x_2
\end{pmatrix}
\]

and
\[
h_{22}(x, \bar{e}_1, \bar{e}_2) = \begin{pmatrix}
-e_3 x_1 \\
e_2 x_1
\end{pmatrix}
\]

So, after putting all above values into eq.(10), we have
\[
F(x,e) = \begin{pmatrix}
B_1 \bar{e}_1 + h_1(x, \bar{e}_1, \bar{e}_2) \\
B_2 \bar{e}_2 + h_{21}(x, \bar{e}_1, \bar{e}_2) + h_{22}(x, \bar{e}_1, \bar{e}_2)
\end{pmatrix},
\]

(11)

Obviously \( \lim_{e_1 \to 0} h_{21}(x, \bar{e}_1, \bar{e}_2) = 0. \)

Now, task is to determine \( \Psi(y,x) \), according to theorem(1) we can define feedback controller as,
\[
\Psi(y,x) = \begin{pmatrix}
\Psi_1(y,x) \\
\Psi_2(y,x)
\end{pmatrix} = \begin{pmatrix}
\Lambda_1 \bar{e}_1 - h_1(x, \bar{e}_1, \bar{e}_2) \\
\Lambda_2 \bar{e}_2 - h_{21}(x, \bar{e}_1, \bar{e}_2)
\end{pmatrix},
\]

(12)

So from equations (11) and (12) error dynamical system (9) can be rewritten as,
\[
\begin{pmatrix}
\frac{d^q e_1}{dt^q} = (B_1 + \Lambda_1)\bar{e}_1 \\
\frac{d^q e_2}{dt^q} = (B_2 + \Lambda_2)\bar{e}_2 + h_{21}(x, \bar{e}_1, \bar{e}_2)
\end{pmatrix}
\]

(13)
So we choose now suitable
\[ B_1 + \Lambda_1 \in \mathbb{R}^1 \] and \[ B_2 + \Lambda_2 \in \mathbb{R}^{2 \times 2} \] which satisfy
\[ |\arg \lambda| \geq q\pi / 2. \]
As Eq.(13) is asymptotically stable with equilibrium point \( e_1 = 0, \overline{e_2} = 0. \)
Obviously \( \lim_{\epsilon_1 \to 0} h_2 (x, \epsilon_1, \overline{e_2}) = 0. \)
This implies the hybrid projective synchronization between response system and master system can be achieved.

VI. NUMERICAL SIMULATIONS

Parameters of the volta system of fractional order are \((a, b, c) = (19, 11, 0.73)\) and for Genesio Tesi system are \((b_1, b_2, b_3, b_4) = (1.1, 1.1, 0.45, 1)\). Fractional order is taken to be \( q = q_1 = q_2 = q_3 = 0.98 \) for which the systems are chaotic. In Eq.(13), we have chosen \( \Lambda_1 = 1 \) and \( \Lambda_2 = \begin{pmatrix} -1 & 0 \\ 0 & c^{-2} \end{pmatrix} \), which leads to stability conditions as eigenvalue of \( B_1 + \Lambda_1 \) is \( \lambda_1 = 2 \) and eigenvalues of \( B_2 + \Lambda_2 \) are \( \lambda_2 = -2, \lambda_3 = -2 \) and for all eigenvalues condition of theorem (1) has been satisfied as \( |\arg \lambda| \geq q\pi / 2. \), where \( q = 0.98 \). The initial conditions for master and slave systems \( [x(0), x_2(0), x_3(0)] = [8.2, 1] \) and \( [y_1(0), y_2(0), y_3(0)] = [0.1, 0.41, 0.31] \) respectively and \( A = \begin{pmatrix} -1 & 0 & -1 \\ 0 & -1 & -2 \\ 1 & 1 & 0.73 \end{pmatrix} \), Then for \( |\epsilon_1(0), e_2(0), e_3(0)| = [899, -4.41, 10, 42] \) and \( Tsim = 20 \); diagram of convergence of errors (see fig.14) is the witness of achieving hybrid projective synchronization between master and slave system.

VII. CONCLUSION

In this article, we have investigated hybrid projective synchronization behavior between two different fractional order Volta and Genesio Tesi systems via tracking control method and stability of fractional order system. Hybrid projective synchronization (HPS) is a more general definition of projective synchronization, in which the drive system and response system could be synchronized up to a vector function factor. HPS is different from the PS and more beneficial to enhance security of communication than any other synchronization because it is obvious that the unpredictability of the vector function factor in HPS is more than that of the same scaling factor in PS. HPS between two different fractional order systems is more useful than that of identical fractional order systems.

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![Fig.14 Error convergence diagram for HPS](image-url)


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