ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 3, Issue 11, May 2014

Analysis of Rectangular Loaded Thin Plate using the Classical Small–Deflection Theory through Variation Iteration Method

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***Abstract*— In the present paper, the differential equation governing on the flow over a thin plate (Bi-harmonic equation) has been investigated analytically by Variational Iteration Method which has many applications in physics and engineering. In this boundary value problem, satisfaction of boundary conditions has great importance and depends only on the nature of the supports structures. Moreover, the results indicate that the mentioned method can be used for solving differential equations governing on this kind of problems with different boundary conditions in a situation that there is no exact solution.**

***Index Terms*— Variational Iteration Method (VIM); Thin Plate; Boundary Value Problems (BVPs); Small–Deflection Theory.**

I. INTRODUCTION

In engineering problems, it is important to know the change of deflection and stress in plates under different loads to design slabs and systems perfectly. In this case, the classical small–deflection theory of thin plates in supplied problems is used to tackle this problem.

Along with rapid development of sciences, many different methods were proposed to solve various boundary–value problems (BVP) [1, 2], such as homotopy perturbation method (HPM)[3-6] and variational iteration method (VIM) [7–13].This paper is devoted to study the behavior of rectangular elastic plate [14, 15 and16] via VIM which is shown in Fig. 1.This method gives rapid convergent successive approximations without any restrictive assumptions or transformation that may change the physical behavior of the problem.

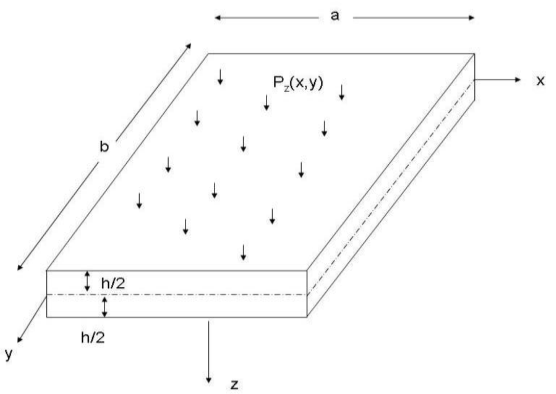

Fig. 1. laterally loaded rectangular plate [14].

Mostly, there are restrictions for the exact analytical solution of plates and there is no general exact solution for different types of boundary condition. Numerical solutions are applied for complex boundary condition and plate shapes to overcome the mentioned weakness.

By using VIM, we successfully find the solution of differential equation governing on the rectangular plates which has excellent agreement with classical solutions.

II. BASIC IDEA OF HE'S VARIATIONAL ITERATION METHOD

To clarify the basic ideas of VIM, we consider the following differential equation:

$$Lu + Nu = g(x,y) \tag{1}$$

Where L is a linear operator, N a nonlinear operator and $g(x,y)$ an inhomogeneous term. According to VIM, we can express respectively, the following correction functional in $x-$ and $y-$ directions, respectively, as follows:

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^x \lambda_1 \left(Lu_n(\tau,y) + N\tilde{u}_n(\tau,y) - g(\tau,y) \right) d\tau \tag{2}$$

$$u_{n+1}(x,y) = u_n(x,y) + \int_0^y \lambda_2 \left(Lu_n(x,\tau) + N\tilde{u}_n(x,\tau) - g(x,\tau) \right) d\tau \tag{3}$$

Where λ_1 and λ_2 are General Lagrange multiplier, which can be identified optimally by the variational theory[17–18].The subscript n indicates the n^{th} approximation and $\tilde{u}_n$ is a restricted variation which means that $\delta \tilde{u}_n = 0$.Consequently, the solution is given by:

$$u(x,y) = \lim_{n\to\infty} u_n(x,y) \tag{4}$$

Now, we will apply the VIM to the following problems to illustrate the strength of the method.

III. MATHEMATICAL MODELING OF THE PROBLEM

The basic differential equation of lateral deflection for thin plates in classical Small-Deflection theory is obtained [14-16]

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
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as follows:

$$\nabla^4 w(x,y)=\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{P(x,y)}{D} \quad (5)$$

Where $w=w(x,y)$ is the middle surface of the plate or the lateral plate displacement, $P(x,y)$ is lateral loads per unit area and D is the plate bending stiffness. In general, The Homogeneous Boundary Conditions (HBCs) for rectangular plate for instance in x-direction) [14, 15] are presented as follows:

Two HBCs for free edge:

$$\begin{cases}\frac{\partial^3 w}{\partial x^3}+(2-\upsilon)\frac{\partial^3 w}{\partial y^2\partial x}=0\\ \frac{\partial^2 w}{\partial x^2}=0\end{cases} \quad (6)$$

For two HBCs for fixed edge:

$$\begin{cases}w=0\\ \frac{\partial w}{\partial x}=0\end{cases} \quad (7)$$

And for two HBCs for simple edge:

$$\begin{cases}w=0\\ -D\left[\frac{\partial^2 w}{\partial x^2}+\upsilon\frac{\partial^2 w}{\partial y^2}\right]=0\end{cases} \quad (8)$$

Where ν is the Poisson ratio of plate material?

It should be noted that two HBCs must be satisfied at $x=x_1$, $x=x_2$ and also the other two HBCs should be satisfied at $y=y_1$, $y=y_2$ .In either case for analysis of plates, we must choose two HBCs for each edge of rectangular plate which depends only on the nature of the plate support.

IV. FORMULATION OF VIM

According to the VIM procedure, we can construct the correction functional for Eq. (5) as follow:

$$w_{n+1}(x,y)=w_n(x,y)+\int_0^x \lambda(\tau)\left(\frac{\partial^4 w_n(\tau,y)}{\partial \tau^4}+2\frac{\partial^4 w_n(\tau,y)}{\partial \tau^2\partial y^2}+\frac{\partial^4 w_n(\tau,y)}{\partial y^4}-\frac{P(\tau,y)}{D}\right)d\tau \quad (9)$$

$$w_{n+1}(x,y)=w_n(x,y)+\int_0^y \lambda(\tau)\left(\frac{\partial^4 w_n(x,\tau)}{\partial x^4}+2\frac{\partial^4 w_n(x,\tau)}{\partial x^2\partial \tau^2}+\frac{\partial^4 w_n(x,\tau)}{\partial \tau^4}-\frac{P(x,\tau)}{D}\right)d\tau \quad (10)$$

Where λ is Lagrange multiplier for $n\geq 0$ which can be identified optimally by the variational theory? According to VIM, the multiplier in $y-$ direction in Eq. (10) yields the following stationary conditions:

$$\lambda^{////}(y)=0,\ \lambda^{///}(y)=-1,\ \lambda^{//}(y)=0,$$
$$\lambda^{/}(y)=0,\ \lambda(y)=0. \quad (11)$$

This in turn gives:

$$\lambda(t)=\frac{(\tau-y)^3}{6} \quad (12)$$

Substituting the value of the Lagrange multiplier, Eq. (12) into the correction functional Eq. (10) gives the following iteration relation:

$$w_{n+1}(x,y)=w_n(x,y)+\int_0^y \frac{(\tau-y)^3}{6}\left(\frac{\partial^4 w_n(x,\tau)}{\partial x^4}+2\frac{\partial^4 w_n(x,\tau)}{\partial x^2\partial \tau^2}+\frac{\partial^4 w_n(x,\tau)}{\partial \tau^4}-\frac{P(x,\tau)}{D}\right)d\tau \quad (13)$$

And the approximation solution is given as $u(x,y)=\lim_{n\to\infty} u_n(x,y)$.

V. NUMERICAL EXAMPLES

In this section, two examples are introduced and the VIM is applied to get the results. The results are then compared with the known exact solutions to ensure the accuracy of the method.

A. *EXAMPLE 1*

For the first case, a rectangular plate with scantling $a\times a$ is considered with constant plate flexural rigidity, D. This plate has two edges which simply is supported and the two edges fixed supported and subjected to a lateral load as follows:

$$P(x,y)=P_0\sin(\frac{\pi x}{a}) \quad (14)$$

The differential equation governing on this case study and the related boundary conditions are presented as follows:

$$\nabla^4 w(x,y)=\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{P_0}{D}\sin(\frac{\pi x}{a}) \quad (15)$$

Then we will have:

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 3, Issue 11, May 2014

$$w(0,y)=0,\quad \frac{\partial^2 w(0,y)}{\partial x^2}=0$$
$$w(a,y)=0,\quad \frac{\partial^2 w(a,y)}{\partial x^2}=0$$
$$w(x,\frac{a}{2})=0,\quad \frac{\partial}{\partial y}w(x,\frac{a}{2})=0$$
$$w(x,-\frac{a}{2})=0,\quad \frac{\partial}{\partial y}w(x,-\frac{a}{2})=0 \tag{16}$$

The exact solution of this problem is reported by Szilard [14]:

$$w(x,y)=\frac{P_0 a^4}{\pi^4 D}\{1+\frac{\sinh(\pi/2)\frac{\pi y}{a}\sinh(\frac{\pi y}{a})}{\pi/2+\cosh(\pi/2)\sinh(\pi/2)}-\frac{\left(\cosh(\frac{\pi y}{a})\right)\left(\frac{\pi}{2}\cosh(\pi/2)+\sinh(\pi/2)\right)}{\pi/2+\cosh(\pi/2)\sinh(\pi/2)}\}\sin(\frac{\pi x}{a}) \tag{17}$$

In order to compare the obtained results with the exact solution, it is necessary to fix the constant values. In this case, the scantling of plate is devoted for comparing the VIM results and the exact solution. For a rectangular plate with scantling 1×1, Eq. (14) and Eq. (17) are changed into these forms:

$$P(x,y)=P_0\sin(\pi x) \tag{18}$$

And

$$w(x,y)=\frac{P_0}{\pi^4 D}\{1+\frac{[(\sinh(\pi y))][(\pi y)\sinh(\pi/2)]}{\pi/2+\cosh(\pi/2)\sinh(\pi/2)}-\frac{\frac{\pi}{2}\cosh(\pi/2)+\sinh(\pi/2)}{\pi/2+\cosh(\pi/2)\sinh(\pi/2)}\cosh(\pi y)\}\sin(\pi x) \tag{19}$$

The variational iteration formula in this case is obtained in the form of:

$$w_{n+1}(x,y)=w_n(x,y)+\int_0^y\frac{(\tau-y)^3}{6}\left(\frac{\partial^4 w_n(x,\tau)}{\partial x^4}+2\frac{\partial^4 w_n(x,\tau)}{\partial x^2\partial\tau^2}+\frac{\partial^4 w_n(x,\tau)}{\partial\tau^4}-\frac{P_0\sin(\pi x)}{D}\right)d\tau \tag{20}$$

Beginning with the initial approximation of $w_0(x,y)$ in the form of a polynomial function as:

$$w_0(x,y)=(\alpha_1+\alpha_2 y+\alpha_3 y^2+\alpha_4 y^3)\sin(\pi x) \tag{21}$$

And α_1, α_2, α_3 and α_4 are unknown constants to be determined with four boundary conditions at $y-$ direction. Using the above iteration formula, Eq. (20), we can directly obtain other components as:

$$w_1(x,y)=\frac{\sin(\pi x)}{2520}[(2520\alpha_2 y+2520\alpha_4 y^3+2520\alpha_3 y^2-7\pi^4\alpha_3 y^6-21\pi^4\alpha_2 y^5+252\pi^2\alpha_4 y^5-105\pi^4\alpha_1 y^4+420\pi^2\alpha_3 y^4-3\pi^4\alpha_4 y^7+2520\alpha_1+105\frac{P_0 y^4}{D}))] \tag{22}$$

In the same manner, the rest of the components of the iteration formula, Eq. (20), can be determine respectively.

Using the boundary conditions, Eq. (16), we obtain the values of four constants of α_1, α_2, α_3 and α_4 as follows:

$$\alpha_1=\frac{0.0015407P_0}{D},\quad \alpha_2=0$$
$$\alpha_3=\frac{-0.0113117P_0}{D},\quad \alpha_4=0 \tag{23}$$

Table1 shows a good agreement between the VIM and the exact solution. The results show that minimum errors occur on both sides in y direction and the error in the middle side is very small and uniform with respect to the exact solution which shows accurate results for VIM.

Table 1: The difference between the exact solution and VIM solutions after four iterations in example 5.1 $\{w(x,y)D/P_0\}$ at x=0.5}.

y	Exact[14]	VIM	Absolute error
-0.5	0.0000000000	2.2740E-12	2.274E-12
-0.4	2.216531E-04	2.2162E-04	2.948E-08
-0.3	6.694487E-04	6.6938E-04	6.135E-08
-0.2	1.116173E-03	1.1160E-03	7.987E-08
-0.1	1.429427E-03	1.4293E-03	8.895E-08
0	1.540853E-03	1.5407E-03	9.164E-08
0.1	1.429427E-03	1.4293E-03	8.895E-08
0.2	1.116173E-03	1.1160E-03	7.987E-08
0.3	6.694487E-04	6.6938E-04	6.135E-08
0.4	2.216531E-04	2.2162E-04	2.948E-08
0.5	4.390851E-12	2.2740E-12	2.274E-12

The 3-D plots of the exact solution and the VIM results are plotted in Fig.2 which indicates the accuracy of VIM.

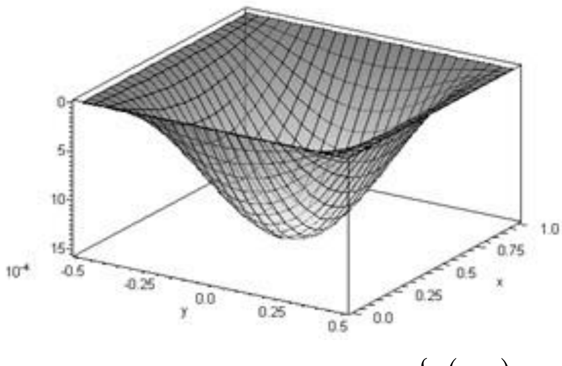

Fig 2.1: The Exact solution of example 5.1 $\{w(x,y)D/P_0\}$

IJEIT
ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
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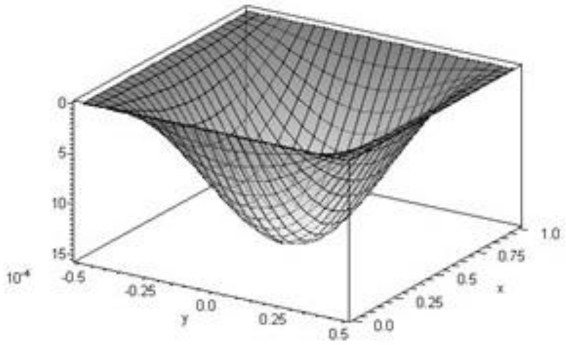

Fig 2.2: The VIM solution of example 5.1 $\{w(x,y)D/P_0\}$

B. EXAMPLE 2

In another case, a rectangular plate with scantling $a\times b$ is considered with constant, D, which is the plate flexural rigidity, and simply supported at all edges and subjected to a lateral load as follows:

$$P(x,y)=P_0\sin(\frac{\pi x}{a})\sin(\frac{\pi y}{b}) \tag{24}$$

The corresponding differential equation and boundary conditions of this case are:

$$\nabla^4 w(x,y)=\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{P_0}{D}\sin(\frac{\pi x}{a})\sin(\frac{\pi y}{b}) \tag{25}$$

And the HBCs are defined as follows:

$$\begin{aligned} &w(0,y)=0, \quad \frac{\partial^2 w(0,y)}{\partial x^2}=0 \\ &w(a,y)=0, \quad \frac{\partial^2 w(a,y)}{\partial x^2}=0 \\ &w(x,0)=0, \quad \frac{\partial^2}{\partial y^2}w(x,0)=0 \\ &w(x,b)=0, \quad \frac{\partial^2}{\partial y^2}w(x,b)=0 \end{aligned} \tag{26}$$

This problem has exact solution as follows [15]:

$$w(x,y)=\frac{P_0}{\pi^4 D(\frac{1}{a^2}+\frac{1}{b^2})^2}\sin(\frac{\pi x}{a})\sin(\frac{\pi y}{b}) \tag{27}$$

Now, we must devote scantling of plate for comparing of result of VIM and exact analysis. For a rectangular plate with scantling 1×1, the Eq. (24) and Eq. (27) change to these forms:

$$P(x,y)=P_0\sin(\pi x)\sin(\pi y) \tag{28}$$

$$w(x,y)=\frac{P_0}{4\pi^4 D}\sin(\pi x)\sin(\pi y) \tag{29}$$

And the variational iteration formula in this case is obtained in the form of:

$$w_{n+1}(x,y)=w_n(x,y)+\int_0^y\frac{(\tau-y)^3}{6}\left(\frac{\partial^4 w_n(x,\tau)}{\partial x^4}+2\frac{\partial^4 w_n(x,\tau)}{\partial x^2\partial\tau^2}+\frac{\partial^4 w_n(x,\tau)}{\partial\tau^4}-\frac{P_0\sin(\pi x)\sin(\pi y)}{D}\right)d\tau \tag{30}$$

Like before, the initial approximation of $w_0(x,y)$ according to the boundary conditions at $x-$ direction, we can make one in the form of a polynomial as:

$$w_0(x,y)=\begin{Bmatrix}\alpha_1+\alpha_2 y+\\ \alpha_3 y^2+\alpha_4 y^3\end{Bmatrix}\sin(\pi x) \tag{31}$$

It is notable that the above formula is depending on the order of equation differentiation. α_1, α_2, α_3 and α_4 are unknown constants to be determined with boundary conditions at $y-$ direction.

By using the above iteration formula, Eq. (30), we can directly obtain other components as:

$$\begin{aligned} w_1(x,y)=\frac{\sin(\pi x)}{2520\pi^4}[&(2520\pi^4\alpha_1+2520\pi^4\alpha_2 y+ \\ &2520\pi^4\alpha_3 y^2+2520\pi^4\alpha_4 y^3-2520\pi\frac{P_0 y}{D}+ \\ &420\pi^3\frac{P_0 y^3}{D}-7\pi^8\alpha_3 y^6-105\pi^8\alpha_1 y^4+ \\ &420\pi^6\alpha_3 y^4-3\pi^8\alpha_4 y^7+252\pi^6\alpha_4 y^5+ \\ &2520\frac{P_0\sin(\pi y)}{D}-21\pi^8\alpha_2 y^5] \end{aligned} \tag{32}$$

In the same manner, the rest of the components of the iteration formula, Eq. (30), can be determined. Finally, the coefficients of α_1, α_2, α_3 and α_4 are introduced as follows:

$$\begin{aligned} &\alpha_1=0 \quad \alpha_2=\frac{0.0080620P_0}{D} \\ &\alpha_3=0 \quad \alpha_4=\frac{-0.0132629P_0}{D} \end{aligned} \tag{33}$$

ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 3, Issue 11, May 2014

Table 2: The difference between the exact solution and VIM solutions after four iterations in example 5.2 $\{w(x,y)D/P_0\}$ at x=0.5}.

Y	EXACT[15]	VIM	ABSOLUTE ERROR
0	0.0000000	0.00000000	0.00E+00
0.1	7.9309 E-04	7.93006E-04	8.418E-08
0.2	1.5085 E-03	1.50838E-03	1.693E-07
0.3	2.0763 E-03	2.07608E-03	2.521E-07
0.4	2.4408 E-03	2.44054E-03	3.324E-07
0.5	2.5664 E-03	2.56609E-03	4.044E-07
0.6	2.4408 E-03	2.44042E-03	4.560E-07
0.7	2.0763 E-03	2.07586E-03	4.710E-07
0.8	1.5085 E-03	1.50812E-03	4.188E-07
0.9	7.9309 E-04	7.92827E-04	2.634E-07
1	−1.052 E-12	4.7298E-09	4.730E-09

The 3-D plots of the exact solution and the VIM results are plotted in Fig.2 which indicates the accuracy of VIM.

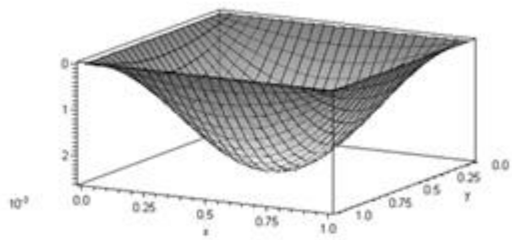

Fig 3.1: The Exact solution of example 5.2 $\{w(x,y)D/P_0\}$

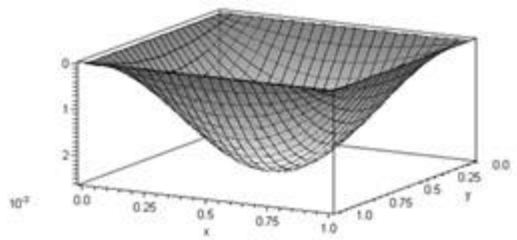

Fig 3.2: The VIM solution of example 5.2 $\{w(x,y)D/P_0\}$

Table 2 shows excellent agreement with the exact solution. Results show that the VIM can be used for certain forms of plates in which loading and the deflection of plate have similar terms in one direction. Therefore; one can integrate it along one direction and simplify the equation. The importance of these terms is to satisfy the boundary conditions along the direction which is not integrated.

VI. CONCLUSIONS

In this paper, the problem of thin plate with the small deflection theory is considered. Two cases of rectangular and square plates analyzed and compared with their own exact classical solutions. The variational iteration method is used to obtain the solution of plate deformation under different loading. Lagrangian multipliers are obtained and used to construct the VIM formulation. The results obtained by VIM are in good agreement with the exact solution. Error calculations show that the VIM method is working very accurate and one can use such methods in engineering problems for the deflection prediction of thin plates.

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ISSN: 2277-3754
ISO 9001:2008 Certified
International Journal of Engineering and Innovative Technology (IJEIT)
Volume 3, Issue 11, May 2014

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